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AD-A218 339

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY
N/A

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE
N/A

3. DISTRIBUTION / AVAILABILITY OF REPORT
Approved for Public Release;
Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
Technical Report No. 274

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 90-0277

6a. NAME OF PERFORMING ORGANIZATION
University of North Carolina
Center for Stochastic Processes

6b. OFFICE SYMBOL *(If applicable)*

7a. NAME OF MONITORING ORGANIZATION
AFOSR/NM

6c. ADDRESS *(City, State, and ZIP Code)*
Statistics Department
CB #3260, Phillips Hall
Chapel Hill, NC 27599-3260

7b. ADDRESS *(City, State, and ZIP Code)*
Bldg. 410
Bolling Air Force Base, DC 20332-6448

8a. NAME OF FUNDING / SPONSORING ORGANIZATION
AFOSR

8b. OFFICE SYMBOL *(If applicable)*
NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
F49620 85C 0144

8c. ADDRESS *(City, State, and ZIP Code)*
Bldg. 410
Bolling AFB, DC 20332-6448

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
6.1102F	2304	A5	

11. TITLE *(Include Security Classification)*
Homogeneous Chaos, p-Forms, Scaling and the Feynman Integral

12. PERSONAL AUTHOR(S)
Johnson, G.W. and Kallianpur, G.

13a. TYPE OF REPORT
preprint

13b. TIME COVERED
FROM ________ TO ______

14. DATE OF REPORT *(Year, Month, Day)*
1989, Sept.

15. PAGE COUNT
67

16. SUPPLEMENTARY NOTATION
None.

17. COSATI CODES

FIELD	GROUP	SUB-GROUP
XXXXX	XXXXXXXXXX	XX

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
None.

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

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20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL
Professor Eytan Barouch

22b. TELEPHONE *(Include Area Code)*
(202)767-~~5026~~ 4940

22c. OFFICE SYMBOL
AFOSR/NM

DD Form 1473, JUN 86 — *Previous editions are obsolete.* — SECURITY CLASSIFICATION OF THIS PAGE

CENTER FOR STOCHASTIC PROCESSES

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HOMOGENEOUS CHAOS, P-FORMS, SCALING
AND THE FEYNMAN INTEGRAL

by

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G. Kallianpur

Technical Report No. 274

September 1989

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Homogeneous Chaos, p-Forms, Scaling and the Feynman Integral

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Dedicated to the Memory of Robert H. Cameron

Acknowledgments:

*To the Center for Stochastic Processes for support and hospitality during my visits from November, 1987 - April, 1988 and in May and June of 1989 and to AFOSR for support through a grant to the Center. Also to U. of Nebraska-Lincoln for support during my sabbatical leave in 1987-88.

**Research supported by the Air Force Office of Scientific Research Contract No. F49620 85C 0144.

1. Introduction. Interesting questions concerning homogeneous chaos, scaling, and the Feynman integral have been brought to light in a recent largely heuristic but fascinating paper of Hu and Meyer [5]. Our purpose here is to indicate a way of resolving these questions as well as others which have arisen in the course of our research.

Let $\mathbb{R}_+$ denote the nonnegative real numbers and let $\mathscr{C}_0 = \mathscr{C}_0(\mathbb{R}_+)$ be the space of continuous functions x on $\mathbb{R}_+$ such that $x(0) = 0$. P_1 will denote the standard Wiener measure on $\mathscr{C}_0(\mathbb{R}_+)$. Every $f \in L^2(\mathscr{C}_0(\mathbb{R}_+), P_1)$ has an expansion in Wiener chaos:

$$f = \sum_{p\geq 0} \frac{1}{p!} I_p(f_p), \tag{1.1}$$

where $f_p \in L_s^2(\mathbb{R}_+^p)$, the symmetric functions which are square integrable over $\mathbb{R}_+^p$, and where I_p denotes the p-fold multiple Wiener-Itô integral.

Hu and Meyer offer the following "formula" in terms of the expansion (1.1):

$$E_\sigma(f) = \sum_k \frac{(\sigma^2-1)^k}{2^k k!} \operatorname{Tr}^k(f_{2k}) . \tag{1.2}$$

The formula (1.2) is to give the "Feynman integral" of the random variable f when σ^2 is purely imaginary and when the right-hand side of (1.2) makes sense.

The first problem coming from the Hu-Meyer paper is that of giving a rigorous treatment of the k-trace, $\operatorname{Tr}^k f_p$, of f_p where $k=0,1,\ldots,[p/2]$ and $[p/2]$ denotes the greatest integer in p/2. We will do this in Section 3, but, for the purpose of this introduction, the reader may think of $\operatorname{Tr}^k f_p$ as given by the (oversimplified) formula

$$(\operatorname{Tr}^k f_p)(s_{2k+1},\ldots,s_p) = \int_{\mathbb{R}_+^k} f_p(s_1,s_1,\ldots,s_k,s_k,s_{2k+1},\ldots,s_p)ds_1\ldots ds_k. \tag{1.3}$$

The difficulties with (1.3) are clear since f_p is defined only up to sets of Lebesgue measure 0.

One would like to connect formula (1.2) with the usual notion of the scalar-valued analytic Feynman integral of f obtained by starting with the Wiener integral

$$\int_{\mathscr{C}_0} f(\sigma x)dP_1(x) \tag{1.4}$$

for $\sigma > 0$ and analytically continuing to σ^2 purely imaginary. Comparing (1.1) and (1.2), the naive hope would be that for $\sigma > 0$,

$$\int_{\mathscr{C}_0} \frac{1}{p!} I_p(f_p)(\sigma x)dP_1(x) = \begin{cases} \frac{(\sigma-1)^k}{2^k k!} \mathrm{Tr}^k(f_{2k}) & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases} \tag{1.5}$$

However, (1.5) is too naive since, for $\sigma \neq 1$, $I_p(f_p)(\sigma x)$ is defined only on a set of P_1-measure 0. (More will be said about this in Section 2A.) In order to obtain a correct version of (1.5), the function $I_p(f_p)$ needs to be extended. Perhaps the first thing that comes to mind is to replace $I_p(f_p)$ in (1.5) with $I_p^\sigma(f_p)$ where $I_p^\sigma(f_p)$ is the Wiener-Itô integral corresponding to the variance parameter σ^2. This does not produce the desired result however because, even though the integral makes sense, we obtain

$$\int_{\mathscr{C}_0} \frac{1}{p!} I_p^\sigma(f_p)(\sigma x)dP_1(x) = 0. \tag{1.6}$$

We will show in this paper how to define $N[I_p(f_p)]$ which we will call the natural extension of the random variable $I_p(f_p)$, and we will obtain the desired formula:

$$\int_{\mathscr{C}_0} \frac{1}{p!} N[I_p(f_p)](\sigma x)dP_1(x) = \begin{cases} \frac{(\sigma-1)^k}{2^k k!} \overrightarrow{\mathrm{Tr}}^k(f_{2k}) & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases} \tag{1.7}$$

$N[I_p(f_p)]$ will be defined in terms of the "scale-invariant $\mathscr{L}^2$-lifting" (to be defined in Section 2C) to random variables on $\mathscr{C}_0(\mathbb{R}_+)$ of certain (p-2k)-forms ($k=0,1,\ldots,[p/2]$) on the Hilbert space $L^2(\mathbb{R}_+)$.

Hu and Meyer have made the suggestion that in studying the problem of extension of $I_p(f_p)$ it might be more natural and basic to start not with Wiener space but with the Cameron-Martin space $\mathscr{H}$ on which the pth order "multiple Wiener integral" is nothing but a homogeneous p-linear form ψ_p defined on $\mathscr{H}$. Since $P_\sigma(\mathscr{H}) = 0$ for all $\sigma > 0$, ψ_p is obviously not a random variable in the usual sense. A theory of "accessible" random variables on a Hilbert space regarded as a finitely additive Gauss probability space has been developed and applied to problems of nonlinear prediction and filtering theory in the recent book by Kallianpur and Karandikar [13]. It turns out that this theory is the appropriate setting for the development of Hu and Meyer's ideas. A key concept is the notion of a lifting map to a suitable representation space, an idea that goes back to I.E. Segal (see the references in [13]). These questions will be taken up in some detail in Section 2C.

It is perhaps worth remarking (although we will not emphasize this point of view in the present paper) that by taking a different choice of representation space, for example, an abstract Wiener space or the white noise space $(\mathscr{S}',\mu)$ where μ is the countably additive Gaussian white noise measure on the space $\mathscr{S}'(\mathbb{R}^d)$ of Schwarz distributions on $\mathbb{R}^d$, one can obtain extensions of our main results to these spaces.

Theorem 5.1 is a key result in our development. It asserts that the p-form $\psi_p(f_p)$ on $L^2(\mathbb{R}_+)$ associated with $f_p \in L^2_s(\mathbb{R}^p_+)$ has a scale-invariant $\mathscr{L}^2$ lifting if and only if the limiting trace, $\overrightarrow{\mathrm{Tr}}^k f_p$, exists for $k = 0,1,\ldots,[p/2]$. Further, it expresses this lifting as a finite sum of multiple Wiener-Itô integrals. The fact that the trace conditions are shown to be necessary as well as sufficient is connected with the definition of the limiting trace,

$\overrightarrow{\mathrm{Tr}}^{k} f_p$ (see Section 3).

The Feynman integral provided the initial motivation for the present work, and it, in conjunction with this paper and the paper of Hu and Meyer [5], suggests several further questions. However, the discussion of the Feynman integral below is limited to issues closely related to those already raised in this introduction.

This paper is addressed primarily to probabilists, but we hope that it will also be of interest to analysts who are concerned with the Feynman integral. With this in mind, the next section on preliminaries is rather detailed. Further, we will use the notation of analysis, in particular, integrals instead of expected values, whenever it seems likely to be clearer to an analyst.

We finish this introduction by outlining the contents of the paper. Section 2 deals with preliminaries; the material is essentially known although some of it is not readily available in the literature and there may be a few novel points. Scaling in Wiener space is reviewed and facts about Wiener-Itô integrals are outlined with special attention paid to scaling. Finally, the scale-invariant lifting of functions on $L^2(\mathbb{R}_+)$ to random variables on Wiener space $C_0(\mathbb{R}_+)$ is defined.

In Section 3, the limiting k-trace, $\overrightarrow{\mathrm{Tr}}^{k} f_p$, is introduced and studied. Section 4 contains two crucial lemmas which give the results of Sections 5 and 6 in the special case where $f_p \in L_s^2(\mathbb{R}_+^p)$ has a finite expansion in terms of a tensorial Hilbert basis $(\phi_{i_1} \otimes \ldots \otimes \phi_{i_p})$ for $L^2(\mathbb{R}_+^p)$.

The key result expressing the lifting of a p-form on $L^2(\mathbb{R}_+)$ as a finite sum of multiple Wiener-Itô integrals or, alternatively, as a p-form on Wiener space is given in Section 5. In Section 6, the point of view is reversed and multiple Wiener-Itô integrals are written as finite sums of liftings of p-forms on $L^2(\mathbb{R}_+)$.

The natural extension $N[I_p(f_p)]$ of $I_p(f_p)$ is defined in Section 7 and we show how to write $N[I_p(f_p)]$ as a finite sum of multiple Wiener-Itô integrals. At this point it is easy to rigorously establish the connection between formula (1.2) of Hu and Meyer and the usual definition of the scalar-valued analytic Feynman integral. This is carried out in Section 8.

During the course of writing this paper, the work of H. Sugita [15] was brought to our attention by S. Watanabe. Sugita's paper and this paper have rather different goals but, among the concerns of [15], are questions similar to the results of Section 5 of our paper.

2. PRELIMINARIES

A. Scaling in Wiener space. A rather detailed treatment of this topic and its relationship with the Feynman integral and other matters as well as references to the earlier literature can be found in the paper [9] of the first author and Skoug. Here we need the basic facts in the setting of Wiener space on the infinite interval $\mathbb{R}_+$. This is the setting of the paper of Hu and Meyer and we follow their discussion for a while.

Given any $\sigma > 0$, let

$$\Omega_\sigma := \{x \in \mathscr{C}_0 \colon [x,x]_t = \sigma^2 t \quad \text{for all dyadic } t,\ t > 0\}$$

where

$$[x,x]_t := \lim_n \sum_{k=0}^{2^n-1} [x(\tfrac{(k+1)t}{2^n}) - x(\tfrac{kt}{2^n})]^2.$$

It is known [5,9] that Wiener measure P_1 on $\mathscr{C}_0(\mathbb{R}_+)$ is carried by Ω_1 and that the scaled measure $P_\sigma := P_1 \circ \sigma^{-1}$ corresponding to the Wiener process with variance parameter σ^2 is carried by Ω_σ. Clearly, $\Omega_{\sigma_1} \cap \Omega_{\sigma_2} = \phi$ if $\sigma_1 \neq \sigma_2$ and so P_{σ_1} and P_{σ_2} are mutually singular. Note: When we say that P_σ is carried by Ω_σ we mean that $P_\sigma(\Omega_\sigma) = 1$ and not that Ω_σ is the topological support of P_σ; indeed, the topological support of P_σ is $\mathscr{C}_0$ for each $\sigma > 0$.

Clearly a function F is defined P_σ- almost surely (P_σ-a.s.) on $\mathscr{C}_0$ if and only if it is defined P_σ-a.s. on Ω_σ. Since $\Omega_\sigma = \sigma\Omega_1$ and $P_\sigma = P_1 \circ \sigma^{-1}$, F is defined P_σ-a.s. if and only if $F \circ \sigma$ is defined P_1-a.s. Thus the Change of Variables Theorem allows us to write

$$\int_{\mathscr{C}_0} F(\sigma x)dP_1(x) = \int_{\mathscr{C}_0} F(y)dP_\sigma(y),$$

or, equivalently,

$$\int_{\Omega_1} F(\sigma x)dP_1(x) = \int_{\Omega_\sigma} F(y)dP_\sigma(y). \tag{2.1}$$

Next, for the convenience of the reader, we state some definitions and results that are given in [9]. Let $\mathcal{B}=\mathcal{B}(\mathcal{C}_0(\mathbb{R}_+))$ denote the σ-algebra of Borel subsets of $\mathcal{C}_0(\mathbb{R}_+)$. When $(\mathcal{C}_0, \mathcal{B}, P_\sigma)$ is completed, let $\mathcal{S}_\sigma$ be the resulting σ-algebra of P_σ-measurable sets and let $\mathcal{N}_\sigma$ be the collection of P_σ-null sets. A subset A of $\mathcal{C}_0$ is said to be <u>scale-invariant measurable</u> provided that $\sigma A \in \mathcal{S}_1$ for all $\sigma > 0$. A scale-invariant measurable set N is said to be <u>scale-invariant null</u> provided $P_1(\sigma N) = 0$ for every $\sigma > 0$. A property which holds except on a scale-invariant null set will be said to hold <u>scale-invariant almost surely or s-a.s.</u> The collection of scale-invariant measurable (respectively, scale-invariant null) sets will be denoted $\mathcal{S}$ (resp., $\mathcal{N}$). In fact, it is easy to show [9, Prop. 3] that $\mathcal{S} = \bigcap_{\sigma>0} \mathcal{S}_\sigma$ and $\mathcal{N} = \bigcap_{\sigma>0} \mathcal{N}_\sigma$, and, further [9, Prop. 4], $A \in \mathcal{S}$ (resp., $A \in \mathcal{N}$) if and only if $A \cap \Omega_\sigma \in \mathcal{S}_\sigma$ (resp., $A \cap \Omega_\sigma \in \mathcal{N}_\sigma$) for every $\sigma > 0$. Theorem 5 of [9] gives a rather helpful characterization of $\mathcal{S}$ and $\mathcal{N}$:

(i) $A \in \mathcal{S}$ if and only if A has the form

$$A = \Big(\bigcup_{\sigma>0} A_\sigma \Big) \cup L$$

where each A_σ is a P_σ-measurable subset of Ω_σ and L is an arbitrary subset of $\mathcal{C}_0 \setminus \bigcup_{\sigma>0} \Omega_\sigma$. Further, for A written as above, $P_\sigma(A) = P_\sigma(A_\sigma)$ for every $\sigma > 0$.

(ii) $N \in \mathcal{N}$ if and only if N has the form

$$N = \Big(\bigcup_{\sigma>0} N_\sigma \Big) \cup L$$

where each N_σ is a P_σ-null subset of Ω_σ and L is an arbitrary subset of $\mathcal{C}_0 \setminus \bigcup_{\sigma>0} \Omega_\sigma$.

A function $F\colon \mathcal{C}_0(\mathbb{R}_+) \longrightarrow \mathbb{R}$ is said to be <u>scale-invariant measurable</u> provided that it is measurable with respect to the σ-algebra $\mathcal{S}$. Every Borel

measurable function and so certainly every continuous function is scale-invariant measurable. Let $F: \mathscr{C}_0 \longrightarrow \mathbb{R}$ have domain D. It is not hard to show [9, Theorem 19] tht F is s-a.s. defined and scale-invariant measurable if and only if, for each $\sigma > 0$, the restriction of F to Ω_σ is P_σ-a.s. defined and P_σ-measurable. Functions F and G from $\mathscr{C}_0(\mathbb{R}_+)$ to $\mathbb{R}$ are said to be equivalent ($F \sim G$) if and only if they are equal s-a.s. This is much more refined than the usual equivalence relation which requires only that F and G be equal P_1-a.s. If the function G is identically 0 on $\mathscr{C}_0$, then it is not surprising that its Feynman integral is 0. It is possible to have a function F such that $F = G$ P_1-a.s. but the Feynman integral of F fails to exist (or, alternately, exists but is not 0). Such examples and others from [9] show the necessity of using the refined equivalence relation in connection with the Feynman integral. On the positive side, if G has an analytic Feynman integral and $F = G$ s-a.s., then F has the same analytic Feynman integral.

<u>B</u>. Multiple Wiener-Itô Integrals. We want to recall one of the ways in which the Wiener stochastic integral $I_1(\phi)$, $\phi \in L^2(\mathbb{R}_+)$, is defined. We will pay special attention to scaling since, except for Section 3, this issue will concern us throughout the rest of this paper. $I_1(\phi)$ is often called the "Wiener integral". We avoid this terminology since, for many, the Wiener integral refers to integration with respect to Wiener measure.

We begin by defining $I_1(\phi)$ for step functions ϕ. Given $t \in (0,+\infty)$, a partition $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0,t]$ and real numbers $c_1, \ldots, c_n$, let

$$\phi(s) := \sum_{i=1}^{n} c_i \chi_{(t_{i-1}, t_i]}(s). \tag{2.2}$$

We define $I_1(\phi): \mathscr{C}_0(\mathbb{R}_+) \longrightarrow \mathbb{R}$ by

$$I_1(\phi)(x) := \sum_{i=1}^{n} c_i [x(t_i) - x(t_{i-1})]. \tag{2.3}$$

Note that for such step functions ϕ, $I_1(\phi)$ is defined on all of $\mathscr{C}_0(\mathbb{R}_+)$ and is also given by the Riemann-Stieltjes integral

$$(2.4) \qquad I_1(\phi)(x) = \int_0^t \phi(s)dx(s) = \int_{\mathbb{R}_+} \phi(s)dx(s).$$

I_1 has the following properties: If ϕ and ψ are step functions and c is a real number, then

(i) $I_1(c\phi) = cI_1(\phi)$;

(ii) $I_1(\phi + \psi) = I_1(\phi) + I_1(\psi)$;

(2.5) (iii) $E[I_1(\phi)] = \int_{\mathscr{C}_0(\mathbb{R}_+)} I_1(\phi)(x)dP_1(x) = 0$;

(iv) $E[|I_1(\phi)|^2] = \|\phi\|_2^2$;

(v) $E[I_1(\phi)I_1(\psi)] = (\phi,\psi)_{L^2(\mathbb{R}_+)}$.

In particular, I_1 is a linear isometry from the vector space of step functions *into* $L^2(\mathscr{C}_0(\mathbb{R}_+),P_1) = L^2(\Omega_1,P_1)$. Since the step functions are dense in $L^2(\mathbb{R}_+)$, I_1 has an extension to all $L^2(\mathbb{R}_+)$ and the extension has properties (i.)-(v.) in (2.5). The following suggestive notation is sometimes used,

$$(2.6) \qquad I_1(\phi)(x) = \int_{\mathbb{R}_+} \phi(s)dx(s),$$

even though the right-hand side of (2.6) cannot be interpreted as an ordinary integral with respect to a function of bounded variation.

For the Wiener process with variance parameter σ, $\sigma > 0$, formulas (2.2)-(2.4) are unchanged but (2.5) becomes

(i') $I_1^\sigma(c\phi) = cI_1^\sigma(\phi)$;

(ii') $I_1^\sigma(\phi + \psi) = I_1^\sigma(\phi) + I_1^\sigma(\psi)$;

(2.7) (iii') $E_{P_\sigma}[I_1^\sigma(\phi)] = \int_{\mathscr{C}_0(\mathbb{R}_+)} I_1^\sigma(\phi)(x)dP_\sigma(x) = 0$;

(iv') $E_{P_\sigma}[|I_1^\sigma(\phi)|^2] = \sigma^2\|\phi\|_2^2$;

(v') $E_{P_\sigma}[I_1^\sigma(\phi)I_1^\sigma(\psi)] = \sigma^2(\phi,\psi)_{L^2(\mathbb{R}_+)}$.

For step functions ϕ, the σ in the notation I_1^σ is not actually necessary. However, for general $\phi \in L^2(\mathbb{R}_+)$, it serves to remind us that $I_1^\sigma(\phi)$ is an element of $L^2(\mathscr{C}_0(\mathbb{R}_+),P_\sigma) = L^2(\Omega_\sigma,P_\sigma)$.

When ϕ is a step function, for every τ, $\sigma > 0$ and $x \in \mathscr{C}_0(\mathbb{R}_+)$, $I_1^\tau(\phi)(\sigma x) = \sigma I_1(\phi)(x)$. (In fact, whether τ is present or not is actually irrelevant in this case.) In particular, for $\sigma > 0$ and $x \in \Omega_1$,

$$I_1^\sigma(\phi)(\sigma x) = \sigma I_1(\phi)(x). \tag{2.8}$$

The Proposition to follow shows that (2.8) can be extended to $\phi \in L^2(\mathbb{R}_+)$.

<u>Proposition 2.1</u> For every $\sigma > 0$ and $\phi \in L^2(\mathbb{R}_+)$ we have

$$I_1^\sigma(\phi)(\sigma x) = \sigma I_1(\phi)(x) \tag{2.9}$$

P_1-a.s.

<u>Proof</u>. Given $\phi \in L^2(\mathbb{R}_+)$, take a sequence (ϕ_n) of step functions such that $\|\phi_n-\phi\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. Then by (i), (ii) and (iv) of (2.5),

$$\begin{aligned} \|\sigma I_1(\phi) - \sigma I_1(\phi_n)\|^2_{L^2(P_1)} &= \sigma^2\|I_1(\phi) - I_1(\phi_n)\|^2_{L^2(P_1)} \\ &= \sigma^2\|\phi-\phi_n\|_2^2 \longrightarrow 0 \end{aligned} \tag{2.10}$$

as $n \longrightarrow \infty$. Further, using (2.8), (2.1) and (i'), (ii') and (iv') of (2.7), we see that

$$\|I_1^\sigma(\phi)\circ\sigma - \sigma I_1(\phi_n)\|_{L^2(P_1)} = \|I_1^\sigma(\phi)\circ\sigma - I_1^\sigma(\phi_n)\circ\sigma\|^2_{L^2(P_1)}$$

$$(2.11) \qquad = \int_{\Omega_1} [I_1^\sigma(\phi)(\sigma x) - I_1^\sigma(\phi_n)(\sigma x)]^2 dP_1(x)$$

$$= \int_{\Omega_1} [I_1^\sigma(\phi)(y) - I_1(\phi_n)(y)]^2 dP_\sigma(y)$$

$$= \sigma^2 \|\phi - \phi_n\|_2^2 \longrightarrow 0$$

as $n \longrightarrow \infty$. Formula (2.9) now follows since (2.10) and (2.11) show that both sides of (2.9) are the limit of the same sequence $(\sigma I_1(\phi_n))$. □

Remark 2.1. When ϕ is a step function, $I_1(\phi)$ is defined on all of $\mathcal{C}_0(\mathbb{R}_+)$ and, in particular, is defined on every Ω_σ, $\sigma > 0$. However, for general $\phi \in L^2(\mathbb{R}_+)$, we only have $I_1(\phi)$ defined P_1-a.s. on Ω_1. $I_1^\sigma(\phi)$ may be regarded as an extension of $I_1(\phi)$ to a function which is defined P_σ-a.s. on Ω_σ. Doing this for every $\sigma > 0$ results in a function which is s-a.s. defined and scale-invariant measurable. Since this is the only extension of $I_1(\phi)$ that we will consider as we continue, we will simply write $I_1(\phi)$ instead of $I_1^\sigma(\phi)$ even when this function is acting on Ω_σ. With this notation, Proposition 2.1 asserts that for every $\phi \in L^2(\mathbb{R}_+)$ and $\sigma > 0$,

$$(2.12) \qquad I_1(\phi)(\sigma x) = \sigma I_1(\phi)(x)$$

P_1-a.s..

The situation will be quite different for $p \geq 2$ and $\phi \in L^2(\mathbb{R}_+^p)$. The multiple Wiener-Itô integral $I_p^\sigma(\phi)$ corresponding to variance parameter σ^2 will be defined P_σ-a.s. on Ω_σ and will provide an extension of $I_p(\phi)$. However, unlike the case $p = 1$, $I_p^\sigma(\phi)$ will not be the only extension of $I_p(\phi)$ of interest to us; we will also be interested in the "natural extension," $N[I_p(\phi)]$. Because of this, we will retain the σ in the notation $I_p^\sigma(\phi)$ when $p \geq 2$.

Remark 2.2. Let $\phi \in L^2(\mathbb{R}_+)$. If desired one can choose a representative

in such a way that $I_1(\phi)$ is everywhere defined on $\mathscr{C}_0(\mathbb{R}_+)$ and satisfies (2.12) for all $\sigma > 0$ and $x \in \mathscr{C}_0(\mathbb{R}_+)$. Simply take $I_1(\phi)$ to be 0 off of $\bigcup_{\sigma>0} \Omega_\sigma$ and on those x in Ω_1 for which $I_1(\phi)$ is not already defined; finally, for $\sigma x \in \Omega_\sigma = \sigma\Omega_1$, take $I_1(\phi)(\sigma x)$ to be $\sigma I_1(\phi)(x)$. Formula (2.12) shows that this everywhere defined function and the original function must agree s-a.s.; that is, P_σ-a.s. for every $\sigma > 0$.

Next we want to discuss the definition and some of the properties of the multiple Wiener-Itô integral $I_p(f)$ where $f \in L^2(\mathbb{R}_+^p)$ and $p \geq 2$. We begin by defining $I_p(f)$ for "special elementary functions" f. Parts of our discussion are adapted from the book of the second author [11, pp. 136-138].

Given $t \in (0,+\infty)$, a partition π: $A_1,\ldots,A_n$ of $[0,t]$ into Borel measurable subsets and a set of real numbers $\{a_{i_1,\ldots,i_p} : i_j = 1,\ldots,n$ for each $j = 1,\ldots,p\}$ such that $a_{i_1,\ldots,i_p} = 0$ when not all of $i_1,\ldots,i_p$ are distinct, let

$$(2.13) \qquad f(s_1,\ldots,s_p) = \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p} \chi_{A_{i_1}\times\ldots\times A_{i_p}}(s_1,\ldots,s_p).$$

Such a function is called a <u>special elementary function</u>. It is an important fact that the set S_p of all special elementary functions is a dense subspace of $L^2(\mathbb{R}_+^p)$.

For $f \in S_p$ and given by (2.13), the multiple Wiener-Itô integral of f is defined P_1-a.s. on $\mathscr{C}_0(\mathbb{R}_+)$ (or on Ω_1) by

$$(2.14) \qquad I_p(f)(x) := \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p} I_1(\chi_{A_{i_1}})(x)\cdot\ldots\cdot I_1(\chi_{A_{i_p}})(x).$$

I_p acting on S_p has the following properties where $f,g \in S_p$ and $c \in \mathbb{R}$:

(i) $\quad I_p(cf) = cI_p(f) \qquad (P_1\text{-a.s.});$

(ii) $I_p(f+g) = I_p(f) + I_p(g) \quad (P_1\text{-a.s.})$;

(iii) $I_p(f) = I_p(\tilde{f}) \quad (P_1\text{-a.s.})$ where $\tilde{f}$ denotes the symmetrization of f;

(2.15) (iv) $E[I_p(f)] = \int_{\mathscr{C}_0(\mathbb{R}_+)} I_p(f)(x)dP_1(x) = 0$;

(v) $$E[|I_p(f)|^2] = E[|I_p(\tilde{f})|^2] = p!\,\|\tilde{f}\|_2^2 \le p!\|f\|_2^2 \;;$$

(vi) $$E[I_p(f)I_p(g)] = E[I_p(\tilde{f})I_p(\tilde{g})] = p!\,(\tilde{f},\tilde{g})_{L^2(\mathbb{R}_+^p)} .$$

From (i), (ii) and (v) of (2.15), we see that I_p <u>has an extension to all of</u> $L^2(\mathbb{R}_+^p)$. In fact, <u>this extension continues to have all of the properties in (2.15)</u>.

Now we turn to the multiple Wiener-Itô integral $I_p^\sigma(f)$ ($\sigma > 0$, $f \in L^2(\mathbb{R}_+^p)$) corresponding to Brownian motion with variance parameter σ^2. It will turn out that $I_p^\sigma(f)$ is defined P_σ-a.s. on Ω_σ.

We know from our earlier discussion of I_1 that for every $\phi \in L^2(\mathbb{R}_+)$ and every $\sigma > 0$, $I_1(\phi)$ is defined P_σ-a.s. on Ω_σ. Hence, for every $f \in S_p$, $I_p^\sigma(f)$ is defined P_σ-a.s. on Ω_σ by the right-hand side of (2.14). Further, from (2.12), for every $\sigma > 0$,

(2.16) $$I_p^\sigma(f)(\sigma x) = \sigma^p I_p(f)(x) \quad (P_1\text{-a.s.}).$$

It can now be shown that I_p^σ acting on S_p has the properties listed in (2.17) below. Further, I_p^σ can be extended to all of $L^2(\mathbb{R}_+^p)$ with the extension continuing to have the properties in (2.17):

(i) $$I_p^\sigma(cf) = cI_p^\sigma(f) \qquad (P_\sigma\text{-a.s.});$$

(ii) $$I_p^\sigma(f+g) = I_p^\sigma(f) + I_p^\sigma(g) \quad (P_\sigma\text{-a.s.});$$

(iii) $$I_p^\sigma(f) = I_p^\sigma(\tilde{f}) \quad (P_\sigma\text{-a.s.});$$

$$(2.17)\quad (\mathrm{iv})\qquad I_p^{\sigma}(f)(\sigma x) = \sigma^p I_p(f)(x) \qquad (P_1\text{-a.s.});$$

$$(\mathrm{v})\qquad E_{P_\sigma}[I_p^{\sigma}(f)] = \int_{\mathcal{C}_0(\mathbb{R}_+)} I_p^{\sigma}(f)(x)dP_\sigma(x) = 0;$$

$$(\mathrm{vi})\qquad E_{P_\sigma}[|I_p^{\sigma}(f)|^2] = E[|I_p^{\sigma}(\tilde{f})|^2]$$

$$= \sigma^{2p}p!\ \|\tilde{f}\|_2^2 \le \sigma^{2p}p!\ \|f\|_2^2;$$

$$(\mathrm{vii})\quad E_{P_\sigma}[I_p^{\sigma}(f)I_p^{\sigma}(g)] = E_{P_\sigma}[I_p^{\sigma}(\tilde{f})I_p^{\sigma}(\tilde{g})]$$

$$= \sigma^{2p}p!\ (\tilde{f},\tilde{g})_{L^2(\mathbb{R}_+^p)}\ .$$

Putting together the functions $I_p^{\sigma}(f)$ on all the Ω_σ's, we obtain an extension of $I_p(f)$ to $\bigcup_{\sigma>0}\Omega_\sigma$ which is s-a.s. defined and scale-invariant measurable. Further, using (2.16), we can employ the same device as in Remark 2.2 and choose a representative which is defined on all of $\mathcal{C}_0(\mathbb{R}_+)$ and satisfies (2.16) for every $\sigma > 0$ and $x \in \mathcal{C}_0(\mathbb{R}_+)$.

C. Lifting and Scale-Invariant Lifting. An extensive discussion of the concept of lifting and its applications to prediction, filtering and smoothing along with references to the literature can be found in the book of Kallianpur and Karandikar [13]. We begin by recalling various facts connected with the canonical Gauss measure on a separable Hilbert space H over $\mathbb{R}$. Let $\mathcal{P}$ denote the class of orthogonal projections on H with finite dimensional range. For $\pi \in \mathcal{P}$, let

$$\mathcal{C}_\pi := \{\pi^{-1}(B) \colon B \in \mathcal{B}(\pi(H)),\ \text{the Borel class of the range of } \pi\}.$$

$\mathcal{C}_\pi$ is a σ-field for each fixed π and $\mathcal{C} := \bigcup_{\pi\in\mathcal{P}} \mathcal{C}_\pi$ is a field of subsets of H. μ will denote the finitely additive canonical Gauss measure on H; i.e., the measure with characteristic function $e^{-\frac{1}{2}\|h\|^2}$ $(h \in H)$ [13, p. 62]. μ is only

finitely additive on $\mathscr{C}$ but is countably additive on $\mathscr{C}_\pi$ for each fixed π. $(H,\mathscr{C},\mu)$ is called a finitely additive canonical Hilbert space.

A representation of μ is a pair (L,P) where P is a countable additive probability measure on some measurable space $(\Omega,\mathscr{A})$ and L is a mapping (or, more precisely, an equivalence class of mappings, see [13, p. 81]) from H into the space of $\mathbb{R}$-valued random variables on $(\Omega,\mathscr{A},P)$ such that L is linear in the following sense:

$$L(a_1h_1 + a_2h_2)(\omega) = a_1L(h_1)(\omega) + a_2L(h_2)(\omega) \quad P\text{-a.s.}$$

for $h_1,h_2 \in H$, $a_1,a_2 \in \mathbb{R}$, and such that for all $C \in \mathscr{C}$,

$$\mu(C) = P\{\omega \in \Omega: (L(h_1)(\omega),\dots,L(h_j)(\omega)) \in B\}$$

where

$$C = \{h \in H: ((h,h_1),\dots,(h,h_j)) \in B\}$$

with $h_1,\dots,h_j$ in H and B a Borel subset of $\mathbb{R}^j$. It is well known that a representation of μ always exists. In the main body of this paper, we will take $(\Omega,\mathscr{A})$ to be $(\mathscr{C}_0(\mathbb{R}_+), \mathscr{B}(\mathscr{C}_0(\mathbb{R}_+)))$ and P to be one of the scaled Wiener measures P_σ, $\sigma > 0$. The representation L will be chosen as

$$L(\phi)(x) = I_1(\phi)(x) \tag{2.18}$$

where $\phi \in H = L^2(\mathbb{R}_+)$ and $x \in \mathscr{C}_0(\mathbb{R}_+)$.

A function $f: H \longrightarrow \mathbb{R}$ is a Borel cylinder function if and only if it can be written as

$$f(h) = g((h,h_1),\dots,(h,h_k)) \tag{2.19}$$

for some $k \geq 1$ and $h_1,\dots,h_k$ in H where $g: \mathbb{R}^k \longrightarrow \mathbb{R}$ is Borel measurable. We define Rf, the lifting of f, to be

$$R(f)(\cdot) := g(I_1(h_1)(\cdot),\dots,I_1(h_k)(\cdot)). \tag{2.20}$$

In light of Proposition 2.1 and Remarks 2.1 and 2.2, we see that Rf is defined s-a.s. (that is, P_σ-a.s. for every $\sigma > 0$).

Let $\sigma > 0$. $\mathcal{L}_\sigma^0(H,\mu)$ will denote the class of functions $f: H \longrightarrow \mathbb{R}$ with the following properties: For all $\pi \in \mathcal{P}$, the function $f\circ\pi(h) := f(\pi h)$ is $\mathcal{C}_\pi$-measurable and for all sequences $\{\pi_N\}$ from $\mathcal{P}$ converging strongly to the identity ($\pi_N \longrightarrow I$), the sequence $\{R(f\circ\pi_N)\}$ is Cauchy in P_σ-probability. Under these circumstances, one can show that all these sequences converge in P_σ-probability to the same limit $R_\sigma(f)$, called the <u>σ-lifting of f</u>. $R_\sigma(f)$ is defined P_σ-a.s. An f in $\mathcal{L}_\sigma^0(H,\mu)$ will be called a σ-accessible random variable. The lifting usually discussed is, in our present terminology, the 1-lifting.

If f has a σ-lifting for all $\sigma > 0$, we let $Rf = R_\sigma f$ on Ω_σ and we call Rf the <u>scale-invariant lifting (or s-lifting)</u> of f. In this case, for every $\sigma > 0$, Rf is defined P_σ-a.s. Thus Rf is s-a.s. defined and scale-invariant measurable. A function f which belongs to $\mathcal{L}_\sigma^0(H,\mu)$ for every $\sigma > 0$ will be called an s-accessible random variable.

For any $\sigma > 0$, we let $\mathcal{L}_\sigma^2(H,\mu)$ denote the set of all $f \in \mathcal{L}_\sigma^0(H,\mu)$ such that for all sequences $\{\pi_N\}$ from $\mathcal{P}$ with $\pi_N \uparrow I$,

$$(2.21) \qquad \int_{\mathcal{C}_0} |R(f\circ\pi_N) - R(f\circ\pi_{N'})|^2 dP_\sigma \longrightarrow 0$$

as $N, N' \longrightarrow \infty$. Note that if $f \in \mathcal{L}_\sigma^2(H,\mu)$, then

$$(2.22) \qquad \int_{\mathcal{C}_0} |R_\sigma(f)|^2 dP_\sigma < \infty.$$

When $f \in \mathcal{L}_\sigma^2(H,\mu)$, we call $R_\sigma(f)$ a <u>σ-$\mathcal{L}^2$ lifting</u>. If f belongs to $\mathcal{L}_\sigma^2(H,\mu)$ for all $\sigma > 0$, we call $Rf := R_\sigma f$ on Ω_σ, a <u>scale-invariant $\mathcal{L}^2$-lifting</u>. If Rf is a scale-invariant $\mathcal{L}^2$-lifting then, for every $\sigma > 0$, Rf is defined P_σ-a.s. and belongs to the space $L^2(\mathcal{C}_0(\mathbb{R}_+),P_\sigma)$ which can be identified with $L^2(\Omega_\sigma,P_\sigma)$.

3. THE LIMITING K-TRACE. There are various possible ways of defining the k-trace of a function $f_p \in L_s^2(\mathbb{R}_+^p)$. However, we focus our attention primarily on the limiting k-trace, $\overrightarrow{Tr}^k f_p$, since it will appear in all of our principal theorems in Sections 5-8. Three other definitions of k-trace will be given. A simple case of the first of these (see Definition 3.1) will be involved in the definition of the limiting trace. The other two will be introduced at the end of this section where it will be shown that, for a large class of functions, all four k-traces exist and agree.

Rosinski discussed a Hilbert space valued trace in [14]. The limiting trace will be defined as the limit of certain simple cases of these traces. We give the definition from [14] just in the setting which concerns us. A somewhat more detailed discussion can be found in our earlier paper [8] and, of course, in [14].

Definition 3.1. Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and let $0 \leq k \leq [p/2]$ where $[p/2]$ denotes the greatest integer in $p/2$. We take $Tr^o f_p := f_p$ to begin with. For $1 \leq k \leq [p/2]$, we say that $Tr^k f_p$ exists and equals $h \in L_s^2(\mathbb{R}^{p-2k})$ if and only if for every CONS (i.e., complete orthonormal set) (e_j) for $L^2(\mathbb{R}_+^k)$,

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} f_p(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{2k}; \cdot, \ldots, \cdot) e_j(s_1, \ldots, s_k) e_j(s_{k+1}, \ldots, s_{2k}) \, ds_1 \ldots ds_k \, ds_{k+1} \ldots ds_{2k} = h(\cdot), \tag{3.1}$$

where the series on the left-hand side of (3.1) converges to h in the norm on $L^2(\mathbb{R}_+^{p-2k})$.

In the main body of this paper, we will need $Tr^k f_p$ as just defined only for the special functions described in our first proposition. Let (ϕ_i) be a CONS for $L^2(\mathbb{R}_+)$ so that $\{\phi_{i_1} \otimes \ldots \otimes \phi_{i_p} : i_\ell = 1, 2, \ldots, \ell = 1, \ldots, p\}$ is a CONS for $L^2(\mathbb{R}_+^p)$.

Proposition 3.1. Suppose that $f_p \in L_s^2(\mathbb{R}_+^p)$ has an expansion of the following form

in terms of the CONS $\{\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}\}$ described above:

$$f_p = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}. \tag{3.2}$$

Then $Tr^k f_p$ exists for every k, $0 \le k \le [p/2]$, and we have

$$Tr^k f_p = \sum_{i_{2k+1},\ldots,i_p=1}^{N}\Big(\sum_{j_1,\ldots,j_k=1}^{N} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p}\Big)\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}. \tag{3.3}$$

<u>Proof</u>. Let (e_j) be any CONS for $L^2(\mathbb{R}_+^k)$. By Definition 3.1 it suffices to show that the series

$$\sum_{j=1}^{\infty}\int_{R_+^k\times R_+^k}\Big[\sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_1}(s_1)\cdot\ldots\cdot\phi_{i_k}(s_k)\phi_{i_{k+1}}(s_{k+1})\cdot\ldots\cdot \phi_{i_{2k}}(s_{2k})\cdot\phi_{i_{2k+1}}(s_{2k+1})\cdot\ldots\cdot\phi_{i_p}(s_p)\Big]\cdot e_j(s_1,\ldots,s_k)e_j(s_{k+1},\ldots,s_{2k})ds_1\cdot\ldots\cdot ds_k\cdot ds_{k+1}\cdot\ldots\cdot ds_{2k} \tag{3.4}$$

converges to the right-hand side of (3.3) in the norm on $L^2(\mathbb{R}_+{}^{p-2k})$. But the series in (3.4) equals

$$\sum_{i_1,\ldots,i_{2k}=1}^{N}\sum_{j=1}^{\infty}\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}\Big) (\phi_{i_1}\otimes\ldots\otimes\phi_{i_k}, e_j)_{L^2(\mathbb{R}_+^k)}\cdot(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}}, e_j)_{L^2(\mathbb{R}_+^k)} \tag{3.5}$$

$$= \sum_{i_1,\ldots,i_{2k}=1}^{N}\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}\Big) \Big(\sum_{j=1}^{\infty}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k}, e_j)(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}}, e_j)\Big)$$

$$= \sum_{i_1,\ldots,i_{2k}=1}^{N}\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}\Big)(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k}, \phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}})$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{N} \Big(\sum_{j_1,\ldots,j_k=1}^{N} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \phi_{i_{2k+1}} \otimes \ldots \otimes \phi_{i_p}$$

with the series in (3.5) converging in the sense of the norm $\|\cdot\|_{L^2(\mathbb{R}_+^{p-2k})}$. The fact that f_p is symmetric implies that the coefficients $a_{i_1,\ldots,i_p}$ are symmetric and this is used in the last equality in (3.5). □

In Proposition 3.6 near the end of this section, we will give a result which is more general than the one which we have just proved. However, Proposition 3.1 is all that we require in our development of the limiting k-trace, our main concern.

Next we take advantage of Proposition 3.1 and define $\overrightarrow{\mathrm{Tr}}^k f_p$.

Definition 3.2. Let $f_p \in L_s^2(\mathbb{R}_+^p)$. Given any CONS (ϕ_i) for $L^2(\mathbb{R}_+)$ and any positive integer N, let

$$(3.6) \qquad f^N_{p,(\phi_i)} = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p} \phi_{i_1} \otimes \ldots \otimes \phi_{i_p}$$

where $a_{i_1,\ldots,i_p} = (f_p, \phi_{i_1} \otimes \ldots \otimes \phi_{i_p})$. For $k=1,\ldots,[p/2]$, the limiting k-trace, $\overrightarrow{\mathrm{Tr}}^k f_p$, exists provided there is a $g \in L^2(\mathbb{R}_+^{p-2k})$ such that

$$(3.7) \qquad \|\mathrm{Tr}^k f^N_{p,(\phi_i)} - g\| \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

for every CONS (ϕ_i) for $L^2(\mathbb{R}_+)$. The function g is, by definition, $\overrightarrow{\mathrm{Tr}}^k f_p$. We take $\overrightarrow{\mathrm{Tr}}^0 f_p := f_p$.

It is natural to ask if $\overrightarrow{\mathrm{Tr}}^k f_p$ exists and equals $\mathrm{Tr}^k f_p$ when f_p has a finite expansion as in Proposition 3.1 above. The answer is in the affirmative. For the purpose of showing this and for some work further on, it will be helpful to state explicitly the following well-known result.

Lemma 3.1. Let (ϕ_i) be a CONS for a separable Hilbert space H over $\mathbb{R}$. For each N, let P_N denote the orthogonal projection onto $sp[\phi_1,\dots,\phi_N]$, the linear span of $\{\phi_1,\dots,\phi_N\}$. Then for any ψ_u, ψ_v in H, we have

$$\sum_{j=1}^{N} (\psi_u,\phi_j)(\psi_v,\phi_j) = (\psi_u, P_N\psi_v). \tag{3.8}$$

Proposition 3.2. Let (ψ_i) be a CONS for $L^2(\mathbb{R}_+)$ and suppose that $f_p \in L^2_s(\mathbb{R}^p_+)$ has the expansion

$$f_p = \sum_{i_1,\dots,i_p=1}^{M} a_{i_1,\dots,i_p} \psi_{i_1} \otimes \dots \otimes \psi_{i_p}. \tag{3.9}$$

Then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists and

$$\overrightarrow{\mathrm{Tr}}^k f_p = \mathrm{Tr}^k f_p. \tag{3.10}$$

Proof. Let (ϕ_i) be any CONS for $L^2(\mathbb{R}_+)$. According to Definition 3.2, we must show that

$$\| \mathrm{Tr}^k f^N_{p,(\phi_i)} - \mathrm{Tr}^k f_p \|_{L^2(\mathbb{R}^{p-2k}_+)} \longrightarrow 0 \tag{3.11}$$

as $N \longrightarrow \infty$. Let $b_{i_1,\dots,i_p} = (f_p, \phi_{i_1} \otimes \dots \otimes \phi_{i_p})$. To show (3.11) it suffices, by Proposition 3.1, to show that

$$\begin{aligned} &\Big\| \sum_{i_{2k+1},\dots,i_p=1}^{N} \Big(\sum_{j_1,\dots,j_k=1}^{N} b_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big) \phi_{i_{2k+1}} \otimes \dots \otimes \phi_{i_p} \\ &- \sum_{i_{2k+1},\dots,i_p=1}^{M} \Big(\sum_{j_1,\dots,j_k=1}^{M} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big) \psi_{i_{2k+1}} \otimes \dots \otimes \psi_{i_p} \Big\| \\ &\longrightarrow 0 \quad \text{as } N \longrightarrow \infty. \end{aligned} \tag{3.12}$$

From the expansion (3.9) we see that

$$b_{i_1,\dots,i_p} = \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p}(\psi_{u_1}\otimes\dots\otimes\psi_{u_p},\phi_{i_1}\otimes\dots\otimes\phi_{i_p})$$

$$(3.13) \qquad = \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p}(\psi_{u_1},\phi_{i_1})\cdot\dots\cdot(\psi_{u_p},\phi_{i_p}).$$

Using (3.13) to justify the first equality and Lemma 3.1 to justify the third, we can write

$$(3.14) \qquad \sum_{i_{2k+1},\dots,i_p=1}^{N}\Big(\sum_{j_1,\dots,j_k=1}^{N} b_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p}\Big)\phi_{i_{2k+1}}\otimes\dots\otimes\phi_{i_p}$$

$$= \sum_{i_{2k+1},\dots,i_p=1}^{N}\left\{\sum_{j_1,\dots,j_k=1}^{N}\left[\sum_{u_1,u_p=1}^{M} a_{u_1,\dots,u_p}(\psi_{u_1},\phi_{j_1})(\psi_{u_2},\phi_{j_1})\cdot\dots\cdot(\psi_{u_{2k-1}},\phi_{j_k})\right.\right.$$

$$\left.\left.\cdot\ (\psi_{u_{2k}},\phi_{j_k})\cdot(\psi_{u_{2k+1}},\phi_{i_{2k+1}})\cdot\dots\cdot(\psi_{u_p},\phi_{i_p})\right]\right\}$$

$$\cdot\Big(\sum_{v_{2k+1},\dots,v_p=1}^{\infty}(\psi_{v_{2k+1}}\otimes\dots\otimes\psi_{v_p},\phi_{i_{2k+1}}\otimes\dots\otimes\phi_{i_p})\psi_{v_{2k+1}}\otimes\dots\otimes\psi_{v_p}\Big)$$

$$= \sum_{v_{2k+1},\dots,v_p=1}^{\infty}\ \sum_{u_1,\dots,u_p=1}^{\infty} a_{u_1,\dots,u_p}\Big(\sum_{j_1=1}^{N}(\psi_{u_1},\phi_{j_1})(\psi_{u_2},\phi_{j_1})\Big)\dots$$

$$\Big(\sum_{j_k=1}^{N}(\psi_{u_{2k-1}},\phi_{j_k})(\psi_{u_{2k}},\phi_{j_k})\Big)\Big(\sum_{i_{2k+1}=1}^{N}(\psi_{u_{2k+1}},\phi_{i_{2k+1}})\cdot(\psi_{v_{2k+1}},\phi_{i_{2k+1}})\Big)\cdot\dots\cdot$$

$$\Big(\sum_{i_p=1}^{N}(\psi_{u_p},\phi_{i_p})(\psi_{v_p},\phi_{i_p})\Big)\psi_{v_{2k+1}}\otimes\dots\otimes\psi_{v_p}$$

$$= \sum_{v_{2k+1},\dots,v_p=1}^{\infty}\ \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p}(\psi_{u_1},P_N\psi_{u_2})\cdot\dots\cdot(\psi_{u_{2k-1}},P_N\psi_{u_{2k}})$$

$$\cdot(\psi_{u_{2k+1}},P_N\psi_{v_{2k+1}})\cdot\dots\cdot(\psi_{u_p},P_N\psi_{v_p})\psi_{v_{2k+1}}\otimes\dots\otimes\psi_{v_p}$$

$$= \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p} (\psi_{u_1}, P_N\psi_{u_2})\cdot\ldots\cdot(\psi_{u_{2k-1}}, P_N\psi_{u_{2k}})\cdot$$

$$\sum_{v_{2k+1},\dots,v_p=1}^{\infty} (P_N\psi_{u_{2k+1}}, \psi_{v_{2k+1}})\cdot\ldots\cdot(P_N\psi_{u_p}, \psi_{v_p})\psi_{v_{2k+1}}\otimes\ldots\otimes\psi_{v_p}$$

$$= \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p} (\psi_{u_1}, P_N\psi_{u_2})\cdot\ldots\cdot(\psi_{u_{2k-1}}, P_N\psi_{u_{2k}})\cdot$$
$$\sum_{v_{2k+1},\dots,v_p=1}^{\infty} (P_N\psi_{u_{2k+1}}\otimes\ldots\otimes P_N\psi_{u_p}, \psi_{v_{2k+1}}\otimes\ldots\otimes\psi_{v_p})\psi_{v_{2k+1}}\otimes\ldots\otimes\psi_{v_p}$$

$$= \sum_{u_1,\dots,u_p=1}^{M} a_{u,\dots,u_p} (\psi_{u_1}, P_N\psi_{u_2})\cdot\ldots\cdot(\psi_{u_{2k-1}}, P_N\psi_{u_{2k}})(P_N\psi_{u_{2k+1}})\otimes\ldots\otimes(P_N\psi_{u_p})\cdot$$

We finish by taking the limit in the norm $\|\cdot\|_{L^2(\mathbb{R}_+^{p-2k})}$ of the last expression in (3.14) thus obtaining

$$(3.15)\quad \lim_{N\to\infty} \sum_{i_{2k+1},\dots,i_p=1}^{N} \Big(\sum_{j_1,\dots,j_k=1}^{N} b_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p}\Big)\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}$$

$$= \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p} \{\lim_{N\to\infty} (\psi_{u_1}, P_N\psi_{u_2})\cdot\ldots\cdot(\psi_{u_{2k-1}}, P_N\psi_{u_{2k}})[P_N\psi_{u_{2k+1}}]\otimes\ldots\otimes[P_N\psi_{u_p}]\}$$

$$= \sum_{u_1,\dots,u_p=1}^{M} a_{u_1,\dots,u_p} (\psi_{u_1}, \psi_{u_2})\cdot\ldots\cdot(\psi_{u_{2k-1}}, \psi_{u_{2k}})\psi_{u_{2k+1}}\otimes\ldots\otimes\psi_{u_p}$$

$$= \sum_{i_{2k+1},\dots,i_p=1}^{M} \Big(\sum_{j_1,\dots,j_k=1}^{M} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p}\Big)\psi_{i_{2k+1}}\otimes\ldots\otimes\psi_{i_p},$$

and so (3.12) is established and the proof is complete. □

<u>Remark 3.1</u>. For the finite sums (3.9), the oversimplified formula (1.3) for the k-trace of f_p does actually give $\mathrm{Tr}^k f_p$ which equals, by Proposition 3.2, $\overrightarrow{\mathrm{Tr}}^k f_p$.

It is desirable to have theorems insuring that $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for a large class of f_p's; Theorem 3.1 is one such result.

<u>Theorem 3.1</u>. Let $f_p \in L^2_s(\mathbb{R}^p_+)$. Suppose that there exists a CONS (ψ_i) for $L^2(\mathbb{R}_+)$ such that, in the expansion

$$(3.16)\qquad f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \psi_{i_1} \otimes \cdots \otimes \psi_{i_p},$$

the coefficients $(a_{i_1,\ldots,i_p})$ are in ℓ_1.

Then for every k, $0 \le k \le [p/2]$, $\overrightarrow{\mathrm{Tr}}^k f_p$ exists and is given by

$$(3.17)\qquad \overrightarrow{\mathrm{Tr}}^k f_p = \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \psi_{i_{2k+1}} \otimes \cdots \otimes \psi_{i_p}.$$

<u>Proof</u>. Because of our assumption on the coefficients, the inner series in (3.17) is absolutely convergent. Further (3.17) is absolutely summable in $i_{2k+1},\ldots,i_p$ in the space $L^2(\mathbb{R}^{p-2k}_+)$ since

$$(3.18)\qquad \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big\| \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \psi_{i_{2k+1}} \otimes \cdots \otimes \psi_{i_p} \Big\|$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big| \sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big|$$

$$\le \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \sum_{j_1,\ldots,j_k=1}^{\infty} \Big| a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big| < \infty.$$

We now see that the right-hand side of (3.17) makes sense and belongs to $L^2(\mathbb{R}^{p-2k}_+)$.

Next let (ϕ_i) be any CONS for $L^2(\mathbb{R}_+)$. To complete the proof, it suffices to show that

$$\lim_{N\to\infty} \sum_{i_{2k+1},\ldots,i_p=1}^{N} \Big(\sum_{j_1,\ldots,j_k=1}^{N} b_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}$$

$$(3.19) \quad = \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \psi_{i_{2k+1}} \otimes \cdots \otimes \psi_{i_p}$$

where

$$(3.20) \quad b_{i_1,\ldots,i_p} = (f_p, \phi_{i_1} \otimes \cdots \otimes \phi_{i_p})$$

$$= \sum_{i_1',\ldots,i_p'=1}^{\infty} a_{i_1',\ldots,i_p'} (\psi_{i_1'} \otimes \cdots \otimes \psi_{i_p'}, \phi_{i_1} \otimes \cdots \otimes \phi_{i_p})$$

and where the limit in (3.19) is in the norm on $L^2(\mathbb{R}_+^{p-2k})$.

Now using (3.20) and the symmetry of the b's to justify the first equality below and Lemma 3.1 to justify the third equality, we can write

$$(3.21) \quad \sum_{i_{2k+1},\ldots,i_p=1}^{N} \Big(\sum_{j_1,\ldots,j_k=1}^{N} b_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{N} \Bigg\{ \sum_{j_1,\ldots,j_k=1}^{N} \Bigg[\sum_{i_1',\ldots,i_p'=1}^{\infty} a_{i_1',\ldots,i_p'} (\psi_{i_1'} \otimes \cdots \otimes \psi_{i_k'}, \phi_{j_1} \otimes \cdots \otimes \phi_{j_k})_{L^2(\mathbb{R}_+^k)}$$

$$\cdot (\psi_{i_{k+1}'} \otimes \cdots \otimes \psi_{i_{2k}'}, \phi_{j_1} \otimes \cdots \otimes \phi_{j_k})_{L^2(\mathbb{R}_+^k)} (\psi_{i_{2k+1}'} \otimes \cdots \otimes \psi_{i_p'}, \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p})_{L^2(\mathbb{R}_+^{p-2k})} \Bigg] \Bigg\}$$

$$\cdot \Big(\sum_{i_{2k+1}'',\ldots,i_p''=1}^{N} (\phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}, \psi_{i_{2k+1}''} \otimes \cdots \otimes \psi_{i_p''}) \psi_{i_{2k+1}''} \otimes \cdots \otimes \psi_{i_p''} \Big)$$

$$= \sum_{i_1',\ldots,i_p'=1}^{\infty} a_{i_1',\ldots,i_p'}$$

$$\sum_{i_{2k+1}'',\ldots,i_p''=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{N} (\psi_{i_1'} \otimes \cdots \otimes \psi_{i_k'}, \phi_{j_1} \otimes \cdots \otimes \phi_{j_k}) (\psi_{i_{k+1}'} \otimes \cdots \otimes \psi_{i_{2k}'}, \phi_{j_1} \otimes \cdots \otimes \phi_{j_k}) \Big)$$

$$\cdot \Big(\sum_{i_{2k+1},\ldots,i_p=1}^{N} (\psi_{i_{2k+1}'} \otimes \cdots \otimes \psi_{i_p'}, \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}) (\psi_{i_{2k+1}''} \otimes \cdots \otimes \psi_{i_p''}) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_p \Big) \Big)$$

$$\cdot \psi_{i_{2k+1}''} \otimes \cdots \otimes \psi_{i_p''}$$

$$= \sum_{i_1',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} \sum_{i_{2k+1}'',\dots,i_p''=1}^{\infty} (\psi_{i_1'} \otimes \dots \otimes \psi_{i_k'}, P_N \psi_{i_{k+1}'} \otimes \dots \otimes \psi_{i_{2k}'})$$

$$\cdot (P_N' \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'}, \psi_{i_{2k+1}''} \otimes \dots \otimes \psi_{i_p''}) \psi_{i_{2k+1}''} \otimes \dots \otimes \psi_{i_p''}$$

where P_N and P_N' are the orthogonal projections onto $sp[(\phi_{j_1} \otimes \dots \otimes \phi_{j_k})_{j_1,\dots,j_k=1}^{N}]$ and $sp[(\phi_{i_{2k+1}} \otimes \dots \otimes \phi_{i_p})_{i_{2k+1},\dots,i_p=1}^{N}]$, respectively.

Beginning with the last term in (3.21), we write

$$(3.22) \quad \sum_{i_1',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} (\psi_{i_1'} \otimes \dots \otimes \psi_{i_k'}, P_N \psi_{i_{k+1}'} \otimes \dots \otimes \psi_{i_{2k}'})$$

$$\cdot \sum_{i_{2k+1}'',\dots,i_p''=1}^{\infty} (P_N' \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'}, \psi_{i_{2k+1}''} \otimes \dots \otimes \psi_{i_p''}) \psi_{i_{2k+1}''} \otimes \dots \otimes \psi_{i_p''}$$

$$= \sum_{i_1',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} (\psi_{i_1'} \otimes \dots \otimes \psi_{i_k'}, P_N \psi_{i_{k+1}'} \otimes \dots \otimes \psi_{i_{2k}'}) P_N' \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'}$$

$$= \sum_{i_1',\dots,i_{2k}'=1}^{\infty} (\psi_{i_1'} \otimes \dots \otimes \psi_{i_k'}, P_N \psi_{i_{k+1}'} \otimes \dots \otimes \psi_{i_{2k}'}) \sum_{i_{2k+1}',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} P_N' \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'}$$

$$= \sum_{i_1',\dots,i_{2k}'=1}^{\infty} (\psi_{i_1'} \otimes \dots \otimes \psi_{i_k'}, P_N \psi_{i_{k+1}'} \otimes \dots \otimes \psi_{i_{2k}'}) P_N' \Big(\sum_{i_{2k+1}',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'} \Big)$$

where the next to last equality is an easy consequence of the Fubini Theorem for the Bochner integral [4, Theorem 3.7.13] and where the last equality is valid since, for each fixed $i_1',\dots,i_{2k}'$,

$$\sum_{i_{2k+1}',\dots,i_p'=1}^{\infty} a_{i_1',\dots,i_p'} \psi_{i_{2k+1}'} \otimes \dots \otimes \psi_{i_p'}$$

is a convergent series in $L^2(\mathbb{R}_+^{p-2k})$.

It remains to use the results of (3.21) and (3.22) to take the limit indicated on the left-hand side of (3.19). We will first carry out the calculations and then

make some comments on their validity.

$$\lim_{N\to\infty} \sum_{i_{2k+1},\ldots,i_p=1}^{N} \Big(\sum_{j_1,\ldots,j_k=1}^{N} b_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p}\Big)\phi_{i_{2k+1}}\otimes\cdots\otimes\phi_{i_p}$$

$$(3.23) = \lim_{N\to\infty} \sum_{i_1,\ldots,i_{2k}=1}^{\infty} (\psi_{i_1}\otimes\cdots\otimes\psi_{i_k}, P_N\psi_{i_{k+1}}\otimes\cdots\otimes\psi_{i_{2k}})$$

$$\cdot P_N'\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p}\Big)$$

$$= \sum_{i_1,\ldots,i_{2k}=1}^{\infty} \lim_{N\to\infty}(\psi_{i_1}\otimes\cdots\otimes\psi_{i_k}, P_N\psi_{i_{k+1}}\otimes\cdots\otimes\psi_{i_{2k}})P_N'\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p}\Big)$$

$$= \sum_{i_1,\ldots,i_{2k}=1}^{\infty} (\psi_{i_1}\otimes\cdots\otimes\psi_{i_k}, \psi_{i_{k+1}}\otimes\cdots\otimes\psi_{i_{2k}}) \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p}$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p}\Big)\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p} .$$

The second equality in (3.23) follows from the Dominated Convergence Theorem for the Bochner integral [4, Theorem 3.7.9] where we are thinking of the sum over $i_1,\ldots,i_{2k}$ as the Bochner integral. The necessary domination holds since

$$\|\psi_{i_1}\otimes\cdots\otimes\psi_{i_k}, P_N\psi_{i_{k+1}}\otimes\cdots\otimes\psi_{i_{2k}})P_N'\Big(\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p}\Big)\|$$

$$\leq \Big\| \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\psi_{i_{2k+1}}\otimes\cdots\otimes\psi_{i_p}\Big\|$$

$$\leq \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} |a_{i_1,\ldots,i_p}|$$

and $\sum_{i_1,\ldots,i_{2k}=1}^{\infty} \Big(\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} |a_{i_1,\ldots,i_p}|\Big) < \infty$. Hence $\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} |a_{i_1,\ldots,i_p}|$ is an integrable dominating function which is independent

of N. The fourth equality in (3.23) follows from the Fubini Theorem for the Bochner integral much as the second equality in (3.22) did. □

The next proposition will give us some information about f_p and $\overrightarrow{\mathrm{Tr}}^k f_p$ under the assumption that $\overrightarrow{\mathrm{Tr}}^k f_p$ exists.

Proposition 3.3. Let $f_p \in L^2_s(\mathbb{R}^p_+)$ and suppose that $\overrightarrow{\mathrm{Tr}}^k f_p$ exists. Then for any CONS (ϕ_i) for $L^2(\mathbb{R}_+)$ and associated expansion

$$(3.24)\qquad f_p = \sum_{i_1,\dots,i_p=1}^{\infty} a_{i_1,\dots,i_p} \phi_{i_1} \otimes \dots \otimes \phi_{i_p}$$

for f_p, we have that the limit

$$(3.25)\qquad \lim_{N\to\infty} \sum_{j_1,\dots,j_k=1}^{N} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p}$$

exists for all $i_{2k+1},\dots,i_p$, and that

$$(3.26)\qquad \sum_{i_{2k+1},\dots,i_p=1}^{\infty} \Big(\sum_{j_1,\dots,j_k=1}^{\infty} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big)^2 < \infty$$

where the inner sum in (3.26) is taken to mean the limit in (3.25). Further, we have the formula

$$(3.27)\qquad \overrightarrow{\mathrm{Tr}}^k f_p = \sum_{i_{2k+1},\dots,i_p=1}^{\infty} \Big(\sum_{j_1,\dots,j_k=1}^{\infty} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big) \phi_{i_{2k+1}} \otimes \dots \otimes \phi_{i_p},$$

where the inner sum in (3.27) is again interpreted as the limit in (3.25) and the outer sum is the limit in the $L^2(\mathbb{R}^{p-2k}_+)$-norm.

Proof. By Definition 3.2 and Proposition 3.1,

$$\mathscr{A}_N := \sum_{i_{2k+1},\dots,i_p=1}^{N} \Big(\sum_{j_1,\dots,j_k=1}^{N} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big) \phi_{i_{2k+1}} \otimes \dots \otimes \phi_{i_p}$$

$$\longrightarrow \overrightarrow{\mathrm{Tr}}^{*} f_p \tag{3.28}$$

in $L^2(\mathbb{R}_+^{p-2k})$-norm as $N \longrightarrow \infty$. Hence for fixed $i'_{2k+1}, \dots, i'_p$,

$$(\mathscr{A}_N, \phi_{i'_{2k+1}} \otimes \dots \otimes \phi_{i'_p}) \longrightarrow (\overrightarrow{\mathrm{Tr}}^{*} f_p, \ \phi_{i'_{2k+1}} \otimes \dots \otimes \phi_{i'_p}). \tag{3.29}$$

But for $N \geq \max\{i'_{2k+1}, \dots, i'_p\}$,

$$(\mathscr{A}_N, \phi_{i'_{2k+1}} \otimes \dots \otimes \phi_{i'_p}) = \sum_{j_1, \dots, j_k=1}^{N} a_{j_1, j_1, \dots, j_k, j_k, i'_{2k+1}, \dots, i'_p}. \tag{3.30}$$

Formula (3.30) and (3.29) show that the limit in (3.25) exists and equals the coefficient of $\phi_{i'_{2k+1}} \otimes \dots \otimes \phi_{i'_p}$ in the expansion of $\overrightarrow{\mathrm{Tr}}^{*} f_p$. Inequality (3.26) and formula (3.27) then follow immediately. □

Proposition 3.3 gives conditions on the coefficients $(a_{i_1, \dots, i_p})$ which are necessary for the existence of $\overrightarrow{\mathrm{Tr}}^{*} f_p$. Are these conditions also sufficient? Let us state this in more detail. Let $(a_{i_1, \dots, i_p})$ be a sequence in ℓ_2 which satisfies (3.25) and (3.26) and let f_p be given by (3.24). Does $\overrightarrow{\mathrm{Tr}}^{*} f_p$ necessarily exist? The answer is "No" as the following example shows.

Let p=2 and take $a_{i_1, i_2} = \begin{cases} \dfrac{(-1)^{i+1}}{i}, & \text{if } i_1 = i_2 = i, \\ 0, & \text{if } i_1 \neq i_2. \end{cases}$

Then $(a_{i_1, i_2}) \in \ell^2$ and

$$\lim_{N \to \infty} \sum_{j_1=1}^{N} a_{j_1, j_1} = \lim_{N \to \infty} \{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{N+1}}{N}\},$$

a limit which exists and is the sum of the alternating harmonic series. The condition (3.26) does not enter into the picture here since p=2. Now let

$f_2 = \sum_{i_1, i_2=1}^{\infty} a_{i_1, i_2} \phi_{i_1} \otimes \phi_{i_2} = \sum_{i=1}^{\infty} \dfrac{(-1)^{i+1}}{i} \phi_i \otimes \phi_i$. To see that $\mathrm{Tr}^k f_p$ does not exist,

take a permutation of the positive integers such that the corresponding rearrangement of the alternating harmonic series has a different sum. Let (ψ_i) be the CONS obtained by making the associated permutation of the ϕ_i's. Let $f_2 = \sum_{i_1,i_2=1}^{\infty} b_{i_1,i_2}\psi_{i_1}\otimes\psi_{i_2}$ be the expansion of f_2 in terms of the CONS $\{\psi_{i_1}\otimes\psi_{i_2}\}$. In order for $\overrightarrow{\mathrm{Tr}}^1 f_2$ to exist, Proposition 3.3 requires that the equality $\lim_{N\to\infty}\sum_{j_1=1}^{N} b_{j_1,j_1} = \lim_{N\to\infty}\sum_{j_1=1}^{N} a_{j_1,j_1}$ hold. However, this equality fails since $\sum_{j_1=1}^{\infty} b_{j_1,j_1}$ is a rearrangement of the alternating harmonic series converging to a different number.

Taking p=2 simplified the discussion of the example above but, in fact, one can also find such examples with $p>2$ where the condition (3.26) does come into consideration.

Next we obtain two results which involve iterated limiting traces.

<u>Definition 3.3</u>. Let $f_p \in L_s^2(\mathbb{R}_+^p)$. If $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\dots,[p/2]$, we say that f_p has all of its first order traces. Whenever $\overrightarrow{\mathrm{Tr}}^k f_p$ exists, it belongs to $L_s^2(\mathbb{R}_+^{p-2k})$ and, for $\nu=0,1,\dots,[(p-2k)/2]$, it may possess a ν-trace $\overrightarrow{\mathrm{Tr}}^\nu[\overrightarrow{\mathrm{Tr}}^k f_p]$. If all of these traces exist, $k=0,1,\dots,[p/2]$, $\nu=0,1,\dots,[(p-2k)/2]$, we say that f_p *has all of its second order traces*. These *second order* (or iterated) *traces are said to be consistent with the first order traces* provided that

$$\overrightarrow{\mathrm{Tr}}^\nu[\overrightarrow{\mathrm{Tr}}^k f_p] = \overrightarrow{\mathrm{Tr}}^{\nu+k} f_p,\ k=0,1,\dots,[p/2],\ \nu=0,1,\dots,[(p-2k)/2].$$

We can, of course, consider third and higher order traces. However, the next simple proposition assures us that we get nothing new beyond the second order.

<u>Proposition 3.4.</u> If $f_p \in L_s^2(\mathbb{R}_+^p)$ has all its first and second order traces and they are consistent, then all the third order traces of f_p also exist and are consistent; that is, if $0\le k\le[p/2]$, $0\le\nu\le[(p-2k)/2]$ and $0\le\ell\le[(p-2k-2\nu)/2]$, then

$\overrightarrow{Tr}^{\ell}\{\overrightarrow{Tr}^{\nu}[\overrightarrow{Tr}^{k} f_p]\}$ exists and

$$\overrightarrow{Tr}^{\ell}\{\overrightarrow{Tr}^{\nu}[\overrightarrow{Tr}^{k} f_p]\} = \overrightarrow{Tr}^{k+\nu+\ell}(f_p). \tag{3.31}$$

Proof. By assumption, $\overrightarrow{Tr}^{\nu}[\overrightarrow{Tr}^{k} f_p]$ exists and $\overrightarrow{Tr}^{\nu}[\overrightarrow{Tr}^{k} f_p] = \overrightarrow{Tr}^{k+\nu} f_p$. Hence,

$$\overrightarrow{Tr}^{\ell}\{\overrightarrow{Tr}^{\nu}[\overrightarrow{Tr}^{k} f_p]\} = \overrightarrow{Tr}^{\ell}\{\overrightarrow{Tr}^{k+\nu} f_p\} = \overrightarrow{Tr}^{k+\nu+\ell} f_p. \qquad \square$$

It may well be that even the second order traces yield nothing new. At any rate, the following proposition is not hard to prove.

Proposition 3.5. Let $f_p \in L_s^2(\mathbb{R}_+^p)$. Suppose that there exists a CONS (ψ_i) for $L^2(\mathbb{R}_+)$ such that, in the expansion

$$f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \psi_{i_1} \otimes \cdots \otimes \psi_{i_p},$$

the coefficients $(a_{i_1,\ldots,i_p})$ are in ℓ_1.

Then f_p has all its first and second order traces and they are consistent.

Proof. By Theorem 3.1, f_p has all its first order traces and, for $0 \le k \le [p/2]$,

$$\overrightarrow{Tr}^{k} f_p = \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \psi_{i_{2k+1}} \otimes \cdots \otimes \psi_{i_p}. \tag{3.32}$$

Since

$$\sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big| \sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big|$$

$$\le \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \; \sum_{j_1,\ldots,j_k=1}^{\infty} \left| a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \right| < +\infty,$$

we see that the coefficients in the expansion (3.32) for $\overrightarrow{Tr}^{k} f_p$ are in ℓ_1. Again applying Theorem 3.1, we have that for all ν, $0 \le \nu \le [(p-2k)/2]$, $\overrightarrow{Tr}^{\nu}(\overrightarrow{Tr}^{k} f_p)$ exists and

$$(3.33)\qquad \overline{\mathrm{Tr}}^{\nu}(\overline{\mathrm{Tr}}^{k} f_p) = \sum_{i_{2(k+\nu)+1},\ldots,i_p=1}^{\infty} \left[\sum_{j_{k+1},\ldots,j_{k+\nu}=1}^{\infty} \cdot \left(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,j_{k+1},j_{k+1},\ldots,j_{k+\nu},j_{k+\nu},i_{2(k+\nu)+1},\ldots,i_p} \right) \right] \cdot \psi_{i_{2(k+\nu)+1}} \otimes \cdots \otimes \psi_{i_p} .$$

But the ℓ_1-assumption on the coefficients $(a_{i_1,\ldots,i_p})$ allows us to use the Fubini Theorem on the inner sums in (3.33) and obtain

$$\overline{\mathrm{Tr}}^{\nu}(\overline{\mathrm{Tr}}^{k} f_p)$$

$$= \sum_{i_{2(k+\nu)+1},\ldots,i_p=1}^{\infty} \left[\sum_{j_1,\ldots,j_{k+\nu}=1}^{\infty} a_{j_1,j_1,\ldots,j_{k+\nu},j_{k+\nu},i_{2(k+\nu)+1},\ldots,i_p} \right] \psi_{i_{2(k+\nu)+1}} \otimes \cdots \otimes \psi_{i_p}$$

which equals, by Theorem 3.1, $\overline{\mathrm{Tr}}^{k+\nu} f_p$. □

We finish this section by giving two further definitions of the k-trace of a function f_p and showing that, under the assumptions of Theorem 3.1 and Proposition 3.5, all four k-traces exist and are equal. Further interesting questions concerning these k-traces and the relationships between them remain to be studied but will not be pursued in this paper.

We begin with the definition of the tensorial k-trace, $\mathrm{Tr}_t^k f_p$.

<u>Definition 3.4.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$. First we take $\mathrm{Tr}_t^0 f_p := f_p$. For $1 \le k \le [p/2]$, $\mathrm{Tr}_t^k f_p$ exists and equals $h \in L_s^2(\mathbb{R}_+^{p-2k})$ if and only if for every k CONSs $(\phi_i^{(j)})$, $j=1,\ldots,k$ for $L^2(\mathbb{R}_+)$ and for every enumeration $e_1, e_2, \ldots$ of the tensorial CONS $(\phi_{i_1}^{(1)} \otimes \cdots \otimes \phi_{i_k}^{(k)})$ for $L^2(\mathbb{R}_+^k)$,

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} f_p(s_1,\ldots,s_k; s_{k+1},\ldots,s_{2k}; \cdot,\ldots,\cdot)$$

$$e_j(s_1,\ldots,s_k) e_j(s_{k+1},\ldots,s_{2k}) ds_1 \ldots ds_k ds_{k+1} \ldots ds_{2k}$$

$$(3.34)\qquad = h(\cdot)$$

where the series on the left-hand side of (3.34) converges to h in the norm on $L^2(\mathbb{R}_+^{p-2k})$.

Comparing Definitions 3.1 and 3.4, it is clear that if $Tr^k f_p$ exists, then $Tr_t^k f_p$ exists and $Tr_t^k f_p = Tr^k f_p$. Further, $Tr_t^1 f_p$ and $Tr^1 f_p$ are exactly the same object since Definitions 3.1 and 3.4 are easily seen to coincide for k=1.

Next we define the iterated k-trace, $Tr_i^k f_p$. This is the definition of k-trace which was given by Hu and Meyer [5].

<u>Definition 3.5.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$. We take $Tr_i^0 f_p = f_p$ and $Tr_i^1 f_p := Tr^1 f_p$ provided that $Tr^1 f_p$ exists. If $Tr^1 f_p$ exists, it belongs to $L_s^2(\mathbb{R}_+^{p-2})$. If this happens and if $Tr^1[Tr^1 f_p]$ exists, we let $Tr_i^2(f_p) := Tr^1(Tr^1 f_p)$. For $1 \leq k \leq [p/2]$, $Tr_i^k f_p$ is defined by iterating this procedure k times

$$Tr_i^k f_p := Tr^1(\cdots(Tr^1(Tr^1 f_p))\cdots),$$

whenever the k successive 1-traces involved all exist.

It is an immediate consequence of the definition that $Tr_i^k f_p$ behaves well with respect to iteration: Let k and υ be nonnegative integers such that $k + \upsilon \leq [p/2]$ and suppose that $Tr_i^{k+\upsilon} f_p$ exists. Then $Tr_i^{\upsilon}(Tr_i^k f_p)$ exists and $Tr_i^{k+\upsilon} f_p = Tr_i^{\upsilon}(Tr_i^k f_p)$.

In Proposition 3.1 we saw that $Tr^k f_p$ exists for functions f_p possessing finite tensorial expansions. The proposition immediately below goes considerably further.

<u>Proposition 3.6.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and suppose that there exists a CONS (ϕ_i) for $L^2(\mathbb{R}_+)$ such that the coefficients $(a_{i_1,\ldots,i_p})$ in the expansion for f_p,

$$f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \phi_{i_1} \otimes \ldots \otimes \phi_{i_p}, \text{ belong to } \ell_1.$$

Then for $0 \leq k \leq [p/2]$, $Tr^k f_p$ exists and is given by

$$(3.35)\ Tr^k f_p = \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \left(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \right) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}.$$

<u>Proof.</u> Let (e_j) be any CONS for $L^2(\mathbb{R}_+^k)$. It suffices to show that

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} f_p(s_1,\ldots,s_k,s_{k+1},\ldots,s_{2k},s_{2k+1},\ldots,s_p)$$

$$e_j(s_1,\ldots,s_k)e_j(s_{k+1},\ldots,s_{2k})ds_1\ldots ds_k ds_{k+1}\ldots ds_{2k}$$

$$(3.36) = \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \left(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \right) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}$$

where the series in j converges in the $L^2(\mathbb{R}_+^{p-2k})$ norm. For the moment, we fix j and consider the integral on the left-hand side of (3.36) with f_p replaced by its series expansion:

$$(3.37) \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} \left[\sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \phi_{i_1}(s_1)\cdot\ldots\cdot\phi_{i_k}(s_k)\cdot\phi_{i_{k+1}}(s_{k+1})\cdot\ldots\cdot\phi_{i_{2k}}(s_{2k})\phi_{i_{2k+1}}\right.$$

$$\left.\cdot (s_{2k+1})\cdot\ldots\cdot\phi_{i_p}(s_p)\right] e_j(s_1,\ldots,s_k)e_j(s_{k+1},\ldots,s_{2k})ds_1\cdot\ldots\cdot ds_k\cdot ds_{k+1}\cdot\ldots\cdot ds_{2k}.$$

Now the sequence of partial sums

$$\sum_{i_1,\ldots,i_p=1}^{N_1,\ldots,N_p} a_{i_1,\ldots,i_p} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}$$

is a sequence of $L_s^2(\mathbb{R}_+^p)$ kernels converging to the $L_s^2(\mathbb{R}_+^p)$ kernel f_p. It follows that the associated integral operators converge in Hilbert-Schmidt norm, hence in operator norm and so certainly in the strong operator topology. Thus the expression in (3.37) equals

$$(3.38) \quad \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} \phi_{i_1}(s_1)\cdot\ldots\cdot\phi_{i_k}(s_k)\cdot e_j(s_1,\ldots,s_k)\phi_{i_{k+1}}(s_{k+1})\cdot\ldots\cdot$$

$$\phi_{i_{2k}}(s_{2k})e_j(s_{k+1},\ldots,s_{2k})\phi_{i_{2k+1}}(s_{2k+1})\cdot\ldots\cdot\phi_{i_p}(s_p)ds_1\ldots ds_k ds_{k+1}\ldots ds_{2k}$$

$$= \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}.$$

Returning to the left-hand side of (3.36) and using our results so far we can write

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}_+^k\times\mathbb{R}_+^k} f_p(s_1,\ldots,s_p)e_j(s_1,\ldots,s_k)e_j(s_{k+1},\ldots,s_{2k})ds_1\ldots ds_{2k}$$

$$(3.39)\quad = \sum_{j=1}^{\infty}\sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}$$

$$= \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\{\sum_{j=1}^{\infty}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)\}\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}$$

$$= \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}})\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p}$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{\infty}\left(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p}\right)\phi_{i_{2k+1}}\otimes\ldots\otimes\phi_{i_p},$$

where the last equality follows from the fact that the coefficients $(a_{i_1,\ldots,i_p})$ are symmetric and in ℓ_1. We will finish the proof by justifying the second equality in (3.39): The sequences $(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)$ and $(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)$ are in ℓ_2 as functions of j. Therefore $|(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)|\ |(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)|$ is in ℓ_1 and, in fact,

$$\sum_{j=1}^{\infty}|(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)|\ |(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)|$$

$$(3.40)\quad \le \left[\sum_{j=1}^{\infty}(\phi_{i_1}\otimes\ldots\otimes\phi_{i_k},e_j)^2\right]^{1/2}\cdot\left[\sum_{j=1}^{\infty}(\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}},e_j)^2\right]^{1/2}$$

$$= \|\phi_{i_1}\otimes\ldots\otimes\phi_{i_k}\|\cdot\|\phi_{i_{k+1}}\otimes\ldots\otimes\phi_{i_{2k}}\|$$

$$= 1.$$

Using (3.40) and the ℓ_1-assumption on the coefficients $(a_{i_1,\dots,i_p})$ and thinking of the sums on both sides of the second equality in (3.39) as Bochner integrals, we can now apply the Fubini Theorem for Bochner integrals [4, Theorem 3.7.13] and obtain the desired equality. □

Corollary 3.1. Under the assumptions of Proposition 3.6, the tensorial trace, $Tr_t^k f_p$, exists and $Tr_t^k f_p = Tr^k f_p$.

Proof. This is trivial since the existence of $Tr^k f_p$ implies the existence of $Tr_t^k f_p$ and their equality. □

Corollary 3.2. Under the assumptions of Proposition 3.6, the iterated trace, $Tr_i^k f_p$, exists and $Tr_i^k f_p = Tr^k f_p$.

Proof. When k=1, the iterated trace and $Tr^1 f_p$ are the same. Hence $Tr_i^1 f_p$ certainly exists and

$$(3.41) \qquad Tr_i^1 f_p = \sum_{i_3,\dots,i_p=1}^{\infty} \left(\sum_{j_1=1}^{\infty} a_{j_1,j_1,i_3,\dots,i_p} \right) \phi_{i_3} \otimes \dots \otimes \phi_{i_p} .$$

Now the coefficients in the expansion (3.41) are again in ℓ_1 as a function of $i_3,\dots,i_p$ since

$$\sum_{i_3,\dots,i_p=1}^{\infty} \left| \sum_{j_1=1}^{\infty} a_{j_1,j_1,i_3,\dots,i_p} \right| \leq \sum_{i_3,\dots,i_p=1}^{\infty} \sum_{j_1=1}^{\infty} |a_{j_1,j_1,i_3,\dots,i_p}| < \infty.$$

Thus $Tr_i^2 f_p := Tr^1[Tr^1 f_p]$ exists and

$$(3.42) \qquad Tr_i^2 f_p = \sum_{i_5,\dots,i_p=1}^{\infty} \left[\sum_{j_2=1}^{\infty} \left(\sum_{j_1=1}^{\infty} a_{j_1,j_1,j_2,j_2,i_5,\dots,i_p} \right) \right] \phi_{i_5} \otimes \dots \otimes \phi_{i_p} .$$

But the ℓ_1-assumption allows us to rewrite the iterated inner sums in (3.42) as a double sum and we obtain

$$(3.43) \quad Tr_i^2 f_p = \sum_{i_5,\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,j_2=1}^{\infty} a_{j_1,j_1,j_2,j_2,i_5,\ldots,i_p} \Big) \phi_{i_5} \otimes \cdots \otimes \phi_{i_p} .$$

Comparing the right-hand side of (3.43) with the right-hand side of (3.35) in the case $k=2$, we see that $Tr_i^2 f_p = Tr^2 f_p$. This argument may be continued to $k=3,\ldots,[p/2]$. □

<u>Corollary 3.3.</u> Under the assumptions of Proposition 3.6 for $0 \leq k \leq [p/2]$, the four k-traces $\overrightarrow{Tr}^k f_p$, $Tr^k f_p$, $Tr_t^k f_p$ and $Tr_i^k f_p$ all exist and are given by the right-hand side of (3.35).

<u>Proof.</u> This follows immediately from the results above and from our earlier result, Theorem 3.1, on the limiting trace. □

4. THE CASE OF FINITE EXPANSIONS. Two of our main results relating multiple Wiener-Itô integrals and liftings of p-forms will be given in this section in the special case where the function f_p involved has a finite tensorial expansion. These results will serve as key lemmas for the next two sections where the corresponding general results will be proved.

We begin by introducing some notation which will be useful in this section as well as further on. Given $\phi_i \in L^2(\mathbb{R}_+)$, let $\xi_i = I_1(\phi_i)$. Formula (2.12) from Section 2B then says that for every $\sigma > 0$,

$$\xi_i(\sigma x) = \sigma \xi_i(x) \quad P_1\text{-a.s.} \tag{4.1}$$

Recalling Remark 2.2, we can even regard ξ_i as everywhere defined on $\mathcal{C}_0(\mathbb{R}_+)$ and satisfying $\xi_i(\sigma x) = \sigma \xi_i(x)$ for all $\sigma > 0$ and $x \in \mathcal{C}_0(\mathbb{R}_+)$.

Given $g \in L^2(\mathbb{R}_+^p)$, there is an associated p-form $\psi_p(g)$ acting on $H = L^2(\mathbb{R}_+^p)$:

$$\begin{aligned} \psi_p(g)(h) &= \int_{\mathbb{R}_+^p} g(s_1,\dots,s_p)h(s_1)\cdot\ldots\cdot h(s_p)ds_1\ldots ds_p \\ &= (g, h^{\otimes p})_{L^2(\mathbb{R}_+^p)} \end{aligned} \tag{4.2}$$

where $h^{\otimes p}(s_1,\dots,s_p) = h(s_1)\cdot\ldots\cdot h(s_p)$. Note that $\xi_i = I_1(\phi_i)$ is the lifting of the 1-form (and cylinder function) $\psi_1(\phi_i)$ defined by $\psi_1(\phi_i)(h) = (\phi_i, h)$. (See (2.19) and (2.20) in Section 2.C.)

Let (ϕ_i) be a CONS for $L^2(\mathbb{R}_+)$ so that $(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p})$ is a CONS for $L^2(\mathbb{R}_+^p)$. We *work throughout the rest of this section* (unless we explicitly say otherwise) *with an* $f_p \in L_s^2(\mathbb{R}_+^p)$ *which has a finite expansion,*

$$f_p = \sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p}\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}. \tag{4.3}$$

It will be convenient for us to explicitly state as a lemma a simple consequence

of the Itô decomposition formula. Itô's decomposition formula, (4.5) below, can be deduced from assertion (3.4) of Theorem 2.2 of [6].

Lemma 4.1. Let $f_p \in L^2_s(\mathbb{R}^p_+)$ have the finite expansion (4.3). Then for any $\sigma > 0$,

$$I^\sigma_p(f_p) = \sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p} I^\sigma_p(\phi_{i_1}\otimes\dots\otimes\phi_{i_p})$$

$$(4.4)\quad = \sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p} I^\sigma_{p-1}(\phi_{i_1}\otimes\dots\otimes\phi_{i_p})\xi_{i_p}$$

$$- \sigma^2(p-1) \sum_{i_1,\dots,i_{p-1}=1}^{N} a_{i_1,i_1,i_2,\dots,i_{p-1}} I^\sigma_{p-2}(\phi_{i_2}\otimes\dots\otimes\phi_{i_{p-1}}).$$

Proof. The first equality in (4.4) is an immediate consequence of the linearity property of I^σ_p on $L^2(\mathbb{R}^p_+)$. The key to the second equality is the Itô decomposition formula mentioned above:

$$I^\sigma_p(\phi_{i_1}\otimes\dots\otimes\phi_{i_p}) = I^\sigma_{p-1}(\phi_{i_1}\otimes\dots\otimes\phi_{i_{p-1}})\xi_{i_p}$$

$$(4.5)\quad - \sigma^2 \sum_{\ell=1}^{p-1} I^\sigma_{p-2}\,(\phi_{i_1}\otimes\dots\otimes\tilde{\phi}_{i_\ell}\otimes\dots\otimes\phi_{i_{p-1}})(\phi_{i_\ell},\phi_{i_p})$$

where the symbol $\sim$ indicates that the ℓth function ϕ_{i_ℓ} is omitted leaving a (p-2)-fold tensor.

Applying (4.5) we obtain

$$\sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} I^\sigma_p(\phi_{i_1}\otimes\dots\otimes\phi_{i_p})$$

$$(4.6)\quad = \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} I^\sigma_{p-1}(\phi_{i_1}\otimes\dots\otimes\phi_{i_{p-1}})\xi_{i_p}$$

$$- \sigma^2 \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p}\{ \sum_{\ell=1}^{p-1} I^\sigma_{p-2}(\phi_{i_1}\otimes\dots\otimes\tilde{\phi}_{i_\ell}\otimes\dots\otimes\phi_{i_{p-1}})(\phi_{i_\ell},\phi_{i_p})\}$$

where here as below we have suppressed the limits of summations on the i_k's all of which range from 1 to N. Note that the first sum on the right-hand side of (4.6) is exactly the first sum on the right-hand side of (4.4). Hence, it suffices to show that the second sum on the right-hand side of (4.6) equals the second sum on the right-hand side of (4.4). Accordingly, using the symmetry of the coefficients to give the second equality below, we can write

$$\sigma^2 \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} \{ \sum_{\ell=1}^{p-1} I^{\sigma}_{p-2}(\phi_{i_1} \otimes \dots \otimes \tilde{\phi}_{i_\ell} \otimes \dots \otimes \phi_{i_{p-1}})(\phi_{i_\ell}, \phi_{i_p})\}$$

$$(4.7) \quad = \sigma^2 \sum_{i_1,\dots,i_{p-1}} a_{i_1,\dots,i_\ell,\dots,i_{p-1},i_\ell} \{ \sum_{\ell=1}^{p-1} I^{\sigma}_{p-2}(\phi_{i_1} \otimes \dots \otimes \tilde{\phi}_{i_\ell} \otimes \dots \otimes \phi_{i_{p-1}})\}$$

$$= \sigma^2 \sum_{\ell=1}^{p-1} \sum_{i_1,\dots,i_{p-1}} a_{i_\ell,i_\ell,i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_{p-1}} I^{\sigma}_{p-2}(\phi_{i_1} \otimes \dots \otimes \check{\phi}_{i_\ell} \otimes \dots \otimes \phi_{i_{p-1}})$$

$$= \sigma^2 \sum_{\ell=1}^{p-1} \sum_{j_1,\dots,j_{p-1}} a_{j_1,j_1,j_2,\dots,j_{p-1}} I^{\sigma}_{p-2}(\phi_{j_2} \otimes \dots \otimes \phi_{j_{p-1}})$$

$$= \sigma^2 (p-1) \sum_{i_1,\dots,i_{p-1}} a_{i_1,i_1,i_2,\dots,i_{p-1}} I^{\sigma}_{p-2}(\phi_{i_2} \otimes \dots \otimes \phi_{i_{p-1}})$$

which establishes the desired expansion. □

We are now ready for the first of the crucial lemmas mentioned in the opening paragraph of this section. The formula involved appears in a paper of Balakrishnan [2, p. 26]. Balakrishnan's proof is tersely written and his result is for "band-limited white noise" in the case $\sigma=1$; nevertheless, the key ideas in our proof appear in Balakrishnan's argument.

Given a positive integer p and a nonnegative integer k such that $0 \leq k \leq [p/2]$, let

$$(4.8) \qquad C_{p,k} := \frac{p!}{(p-2k)!2^k k!} .$$

These constants will appear frequently throughout the rest of the paper.

Lemma 4.2. Let $f_p \in L^2_s(\mathbb{R}^p_+)$ be given by the finite expansion (4.3). Then, for any $\sigma > 0$, we have

$$(4.9) \qquad I^\sigma_p(f_p) = \sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p} I^\sigma_p(\phi_{i_1}\otimes\dots\otimes\phi_{i_p})$$

$$= \sum_{k=0}^{[p/2]} (-1)^k \sigma^{2k} C_{p,k} \sum_{i_{2k+1},\dots,i_p=1}^{N} \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p}$$

$$\cdot\, \xi_{i_{2k+1}}\cdot\ldots\cdot\xi_{i_p}$$

$$= \sum_{k=0}^{[p/2]} (-1)^k \sigma^{2k} C_{p,k}\, R[\psi_{p-2k}(\mathrm{Tr}^k f_p)]$$

Proof. The subscripts on the a's range from 1 to N throughout the proof, but, for purposes of simplification, we will suppress this range in the notation.

The first equality in (4.9) comes simply from (4.3) and the linearity of I^σ_p acting on $L^2(\mathbb{R}^p_+)$. The fourth equality comes from formula (2.20) for lifting cylinder functions, (4.2) above and the fact (see Propositions 3.1 and 3.2) that

$$(4.10) \quad \mathrm{Tr}^k f_p = \sum_{i_{2k+1},\dots,i_p} \sum_{i_1,\dots,i_k} a_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p} \phi_{i_{2k+1}}\otimes\dots\otimes\phi_{i_p}.$$

Our main task then is to establish the second equality in (4.9).

We proceed by induction on p. When p=1, the sum over k reduces to the k=0 term. Further, when k=0, the a's do not have any repeated indices. Thus we see that when p=1, the third expression in (4.9) is just another way of writing the second expression.

The case p=2 is readily obtained from (4.4) of Lemma 4.1:

$$\sum_{i_1,i_2} a_{i_1,i_2} I^\sigma_2(\phi_{i_1}\otimes\phi_{i_2}) = \sum_{i_1,i_2} a_{i_1,i_2} I^\sigma_2(\phi_{i_1})\xi_{i_2} - \sigma^2 \sum_{i_1} a_{i_1,i_1}$$

$$= \sum_{i_1, i_2} a_{i_1, i_2} \xi_{i_1} \xi_{i_2} - \sigma^2 \sum_{i_1} a_{i_1, i_1}$$

where the second equality comes from the fact that $I_1^\sigma(\phi_{i_1}) = \xi_{i_1}$ for every $\sigma > 0$.

Next we consider an arbitrary integer p making the induction assumption that (4.9) holds for all integers less than p. (Actually, as pointed out above, we need only concern ourselves with the second equality in (4.9).) In the string of four equalities immediately below, the first follows from Lemma 4.1, the third from the induction hypothesis, and the other two come from interchanging orders of summation. Since the sums are finite, there is no question about the validity of the interchange of the orders of summation.

$$(4.11) \quad \sum_{i_1, \ldots, i_p} a_{i_1, \ldots, i_p} I_p^\sigma(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}) = \sum_{i_1, \ldots, i_p} a_{i_1, \ldots, i_1} I_{p-1}^\sigma(\phi_{i_1} \otimes \cdots \otimes \phi_{i_{p-1}}) \xi_{i_p}$$

$$- (p-1)\sigma^2 \sum_{i_1, \ldots, i_{p-1}} a_{i_1, i_1, i_2, \ldots, i_{p-1}} I_{p-2}^\sigma(\phi_{i_2} \otimes \cdots \otimes \phi_{i_{p-1}})$$

$$= \sum_{i_p} \{ \sum_{i_1, \ldots, i_{p-1}} a_{i_1, \ldots, i_{p-1}, i_p} I_{p-1}^\sigma(\phi_{i_1} \otimes \cdots \otimes \phi_{i_{p-1}}) \} \xi_{i_p}$$

$$- (p-1)\sigma^2 \sum_{i_2, \ldots, i_{p-1}} (\sum_{i_1} a_{i_1, i_1, i_2, \ldots, i_{p-1}}) I_{p-2}^\sigma(\phi_{i_2} \otimes \cdots \otimes \phi_{i_{p-1}})$$

$$= \sum_{i_p} \{ \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \frac{(p-1)!(-1)^k \sigma^{2k}}{(p-1-2k)! 2^k k!} \sum_{i_{2k+1}, \ldots, i_p} a_{i_1, i_1, \ldots, i_k, i_k, i_{2k+1}, \ldots, i_{p-1}, i_p}$$

$$\cdot \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_{p-1}} \} \xi_{i_p}$$

$$-(p-1) \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \frac{(p-2)!(-1)^k \sigma^{2(k+1)}}{(p-2-2k)! 2^k k!} \sum_{j_{2k+1}, \ldots, j_{p-2}} \sum_{j_1, \ldots, j_k} \sum_{i_1}$$

$$a_{i_1,i_1,j_1,j_1,\dots,j_k,j_k,j_{2k+1},\dots,j_{p-2}}\,\xi_{j_{2k+1}}\cdot\ldots\cdot\xi_{j_{p-2}}$$

$$= \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \frac{(p-1)!(-1)^k\sigma^{2k}}{(p-1-2k)!2^k k!} \sum_{i_{2k+1},\dots,i_p} \sum_{i_1,\dots,i_k} a_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p}$$

$$\cdot\ \xi_{i_{2k+1}}\cdot\ldots\cdot\xi_{i_p}$$

$$+ \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \frac{(p-1)!(-1)^k\sigma^{2(k+1)}}{(p-2-2k)!2^k k!} \sum_{j_{2k+1},\dots,j_{p-2}}$$

$$\sum_{i_1,j_1,\dots,j_k} a_{i_1,i_1,j_1,j_1,\dots,j_k,j_k,j_{2k+1},\dots,j_{p-2}}\,\xi_{j_{2k+1}}\cdot\ldots\cdot\xi_{j_{p-2}}.$$

Next we work with the second of the two expressions on the right-hand side of the last equality in (4.11); first we rename the dummy indices of summation, then we shift the sum over k by 1 and, finally, we rename the indices again:

$$\sum_{k=0}^{\left[\frac{p-2}{2}\right]} \frac{(p-1)!(-1)^{k+1}\sigma^{2(k+1)}}{(p-2-2k)!2^k k!} \sum_{j_{2k+1},\dots,j_{p-2}}$$

$$\sum_{i_1,j_1,\dots,j_k} a_{i_1,i_1,j_1,j_1,\dots,j_k,j_k,j_{2k+1},\dots,j_{p-2}}\,\xi_{j_{2k+1}}\cdot\ldots\cdot\xi_{j_{p-2}}$$

$$(4.12) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \frac{(p-1)!(-1)^{k+1}\sigma^{2(k+1)}}{(p-2-2k)!2^k k!} \sum_{\ell_{2k+3},\dots,\ell_p}$$

$$\sum_{\ell_1,\dots,\ell_{k+1}} a_{\ell_1,\ell_1,\dots,\ell_{k+1},\ell_{k+1},\ell_{2k+3},\dots,\ell_p}\,\xi_{\ell_{2k+3}}\cdot\ldots\cdot\xi_{\ell_p}$$

$$= \sum_{k'=1}^{\left[\frac{p}{2}\right]} \frac{(p-1)!(-1)^{k'}\sigma^{2k'}}{[(p-2-2(k'-1)]!2^{k'-1}(k'-1)!} \sum_{\ell_{2k'+1},\dots,\ell_p}$$

$$\sum_{\ell_1,\dots,\ell_{k'}} a_{\ell_1,\ell_1,\dots,\ell_{k'},\ell_{k'},\ell_{2k'+1},\dots,\ell_p}\,\xi_{\ell_{2k'+1}}\cdot\ldots\cdot\xi_{\ell_p}$$

$$= \sum_{k=1}^{\left[\frac{p}{2}\right]} \frac{(p-1)!(-1)^k \sigma^{2k}}{(p-2k)!2^{k-1}(k-1)!} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p} .$$

Next we substitute the result of (4.12) for the second expression on the right-hand side of the last equality in (4.11); then we combine the sums to obtain (4.9) as desired. We will carry out the combining of sums under the assumption that p is odd in which case $[\frac{p-1}{2}] = [\frac{p}{2}]$. Finally, we will finish the proof by noting the adjustments that must be made for the case of even p.

(4.13) $$\sum_{i_1,\ldots,i_p} a_{i_1,\ldots,i_p} I_p^{\sigma}(\phi_{i_1} \otimes \ldots \otimes \phi_{i_p})$$

$$= \sum_{k=0}^{\left[\frac{p}{2}\right]} \frac{(p-1)!(-1)^k \sigma^{2k}}{(p-1-2k)!2^k k!} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

$$+ \sum_{k=1}^{\left[\frac{p}{2}\right]} \frac{(p-1)!(-1)^k \sigma^{2k}}{(p-2k)!2^{k-1}(k-1)!} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1 i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

$$= \sum_{i_1,\ldots,i_p} a_{i_1,\ldots,i_p} \xi_{i_1} \cdot \ldots \cdot \xi_{i_p}$$

$$+ \sum_{k=1}^{\left[\frac{p}{2}\right]} \frac{(p-1)!(-1)^k \sigma^{2k}}{(p-1-2k)!2^{k-1}(k-1)!} \left\{\frac{1}{2k} + \frac{1}{p-2k}\right\} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

$$= \sum_{i_1,\ldots,i_p} a_{i_1,\ldots,i_p} \xi_{i_1} \cdot \ldots \cdot \xi_{i_p}$$

$$+ \sum_{k=1}^{\left[\frac{p}{2}\right]} \frac{p!(-1)^k \sigma^{2k}}{(p-2k)!2^k k!} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

$$= \sum_{k=0}^{\left[\frac{p}{2}\right]} (-1)^k \sigma^{2k} C_{p,k} \sum_{i_{2k+1},\ldots,i_p}$$

$$\sum_{i_1,\ldots,i_k} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

which gives the second equality in (4.9) as desired except that we need to make some comments about the case when p is even.

When p is even, $[\frac{p}{2}] = [\frac{p-1}{2}] + 1$. The terms from k=1 to $k = [\frac{p-1}{2}]$ can be added just as in the case of odd p with the same result. However, the second series on the right-hand side of the first equality in (4.13) will have a $k = [\frac{p}{2}] = \frac{p}{2}$ term which is not present in the first series. The term is just

$$\frac{(p-1)!(-1)^{p/2}\sigma^p}{0!2^{\frac{p}{2}-1}(\frac{p}{2}-1)!} \sum_{i_1,\ldots,i_{p/2}} a_{i_1,i_1,\ldots,i_{p/2},i_{p/2}}$$

$$= \frac{p!(-1)^{p/2}\sigma^p}{2(p/2)2^{\frac{p}{2}-1}(\frac{p}{2}-1)!} \sum_{i_1,\ldots,i_{p/2}} a_{i_1,i_1,\ldots,i_{p/2},i_{p/2}}$$

$$= \frac{p!(-1)^{p/2}\sigma^p}{0!2^{\frac{p}{2}}(\frac{p}{2})!} \sum_{i_1,\ldots,i_{p/2}} a_{i_1,i_1,\ldots,i_{p/2},i_{p/2}}$$

$$= (-1)^{p/2}\sigma^p C_{p,p/2} \sum_{i_1,\ldots,i_{p/2}} a_{i_1,i_1,\ldots,i_{p/2},i_{p/2}}$$

which is precisely the $k = [p/2]$ term on the right-hand side of the second equality in (4.9) and so the proof is complete. □

Remark 4.1. Lemma 4.2 makes the idea of the "natural extension" of $I_p(f_p) = I_p^1(f_p)$ rather transparent in the case of f_p's with finite expansions. Hu and Meyer [5] sought to extend $I_p(f_p)$ in such a way as to preserve polynomials in first order stochastic integrals. That this is a reasonable strategy in connection with the Feynman integral was quite believable to the present authors since it is consistent with their earlier work, for example [7 or 12], relating the Fresnel integral of Albeverio and Hoegh-Krohn [1] which deals with certain functions on a Hilbert space H to the Fresnel (or Feynman) integral of corresponding functions on the Wiener space [3,10] or abstract Wiener space [12] associated with H.

Applying Lemma 4.2 with $\sigma = 1$, we get the formulas

$$I_p(f_p) = \sum_{k=0}^{[\frac{p}{2}]} (-1)^k C_{p,k} \sum_{i_{2k+1},\dots,i_p=1}^{N} \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p} \cdot \xi_{i_{2k+1}} \cdot\ldots\cdot \xi_{i_p} \tag{4.14}$$

$$= \sum_{k=0}^{[\frac{p}{2}]} (-1)^k C_{p,k}\, R[\psi_{p-2k}(\mathrm{Tr}^k f_p)]\ .$$

Since the natural extension $N[I_p(f_p)]$ is to preserve polynomials in first order stochastic integrals, that is, in the ξ_i's, it is clear how $N[I_p(f_p)]$ should be defined in this case; it should continue to be given by either of the last two expressions in (4.14):

$$N[I_p(f_p)] = \sum_{k=0}^{[\frac{p}{2}]} (-1)^k C_{p,k} \sum_{i_{2k+1},\dots,i_p=1}^{N} \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,i_1,\dots,i_k,i_k,i_{2k+1},\dots,i_p} \cdot \xi_{i_{2k+1}} \cdot\ldots\cdot \xi_{i_p} \tag{4.15}$$

$$= \sum_{k=0}^{[\frac{p}{2}]} (-1)^k C_{p,k} \, R[\psi_{p-2k}(Tr^k f_p)].$$

Since each ξ_i is s-a.s. defined and may even be taken as everywhere defined, the same can be said of $N[I_p(f_p)]$ in the present situation.

In the general case in Section 7 where infinite expansions are involved, we will need to work harder to define the natural extension. Briefly, for every $\sigma > 0$, we will take limits in the space $L^2(\Omega_\sigma, P_\sigma)$ of polynomials in first order stochastic integrals.

In Lemma 4.2 a multiple Wiener-Itô integral $I_p^\sigma(f_p)$ was expressed as a sum of (p-2k)-forms, k=0,1,...,[p/2]. Lemma 4.3 will express a p-form, restricted to Ω_σ, as a sum of multiple Wiener-Itô integrals $I_{p-2k}^\sigma(Tr^k f_p)$, k=0,1,...,[p/2]. Lemma 4.2 will be the key to the proof of Lemma 4.3 which will, in turn, be essential to the developments in later sections. Formula (4.18) below is essentially formula (10) in Hu and Meyer [5], but in [5], it is a remark which is not pursued.

Again f_p will be assumed to have the finite expansion (4.3). Hence, the associated p-form $\psi_p(f_p)$ on $H=L^2(\mathbb{R}_+)$ (see (4.2)) will be the cylinder function

$$\psi_p(f_p)(h) = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p} (\phi_{i_1},h)\cdot\ldots\cdot(\phi_{i_p},h). \tag{4.16}$$

Hence $\psi_p(f_p)$ certainly has a lifting,

$$\Psi_p(x) = R[\psi_p(f_p)](x) = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p} \xi_{i_1}(x)\cdot\ldots\cdot\xi_{i_p}(x), \tag{4.17}$$

which is a p-form on $\mathscr{C}_0(\mathbb{R}_+)$ which may be regarded as everywhere defined.

<u>Lemma 4.3.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$ be given by the finite expansion (4.3) and let the associated p-form $\Psi_p = R[\psi_p(f_p)]$ be given on $\mathscr{C}_0(\mathbb{R}_+)$ by (4.17). Then s-a.s.

(4.18) $$\Psi_p = R[\psi_p(f_p)] = \sum_{k=0}^{[\frac{p}{2}]} \sigma^{2k} C_{p,k} I^{\sigma}_{p-2k}(\mathrm{Tr}^k f_p).$$

<u>Proof:</u> The various indices i_j in the proof vary from 1 to N as in (4.17) but we suppress this fact below. We proceed by induction on p. When p=1, the last expression in (4.18) reduces to $I_1(\sum_{i_1} a_{i_1} \phi_{i_1})$ which equals Ψ_1 as required. The case p=2 is easily obtained from Lemma 4.2:

$$I^{\sigma}_2(f_2) = \sum_{i_1,i_2} a_{i_1,i_2} \xi_{i_1} \xi_{i_2} - \sigma^2 \sum_{i_1} a_{i_1,i_1}$$

$$= \Psi_2 - \sigma^2 \sum_{i_1} a_{i_1,i_1} .$$

Therefore, $\Psi_2 = I^{\sigma}_2(f_2) + \sigma^2 \mathrm{Tr}^1(f_2)$ as desired.

We now assume that (4.18) holds for all integers less than p and examine the pth case itself. The k=0 (υ=0 in our present notation) term in the third expression in (4.9) is just Ψ_p. This yields the first of the two equalities that follow:

$$\Psi_p = I^{\sigma}_p(f_p) + \sum_{\upsilon=1}^{[p/2]} (-1)^{\upsilon+1} \sigma^{2\upsilon} C_{p,\upsilon} \sum_{i_{2\upsilon+1},\ldots,i_p} \Big(\sum_{i_1,\ldots,i_\upsilon} a_{i_1,i_1,\ldots,i_\upsilon,i_\upsilon,i_{2\upsilon+1},\ldots,i_p} \Big) \cdot \xi_{i_{2\upsilon+1}} \cdot \ldots \cdot \xi_{i_p}$$

(4.19) $$= I^{\sigma}_p(f_p) + \sum_{\upsilon=1}^{[p/2]} (-1)^{\upsilon+1} \sigma^{2\upsilon} C_{p,\upsilon} \Big\{ \sum_{i_{2\upsilon+1},\ldots,i_p} c_{i_{2\upsilon+1},\ldots,i_p} \xi_{i_{2\upsilon+1}} \cdot \ldots \cdot \xi_{i_p} \Big\}$$

where $c_{i_{2\upsilon+1},\ldots,i_p} = \sum_{i_1,\ldots,i_\upsilon} a_{i_1,i_1,\ldots,i_\upsilon,i_\upsilon,i_{2\upsilon+1},\ldots,i_p}$.

Now we can apply the induction hypothesis to the inner sum in the last expression of (4.19) since $\upsilon \geq 1$. Doing this and letting

(4.20) $$h_{p-2\upsilon} = \sum_{i_{2\upsilon+1},\ldots,i_p} c_{i_{2\upsilon+1},\ldots,i_p} \phi_{i_{2\upsilon+1}} \otimes \ldots \otimes \phi_{i_p} .$$

we obtain

$$(4.21)\ \Psi_p = I_p^\sigma(f_p) + \sum_{\nu=1}^{[\frac{p}{2}]} \frac{p!(-1)^{\nu+1}\sigma^{2\nu}}{(p-2\gamma)!2^\nu \nu!} \{ \sum_{r=0}^{\frac{[p-2\nu]}{2}} \frac{p-2\nu)!\sigma^{2r}}{(p-2\nu-2r)!2^r r!} I^\sigma_{p-2\nu-2r}(Tr^r h_{p-2\nu})\}.$$

Now $Tr^r(h_{p-2\nu}) = Tr^r[Tr^\nu f_p] = Tr^{r+\nu} f_p$ where the second equality follows from Proposition 3.5. Further, applying the Binomial Theorem to $(-1+1)^k$, one sees that

$$\sum \frac{(-1)^{\nu+1}}{\nu!r!} = \frac{1}{k!}$$

where this last sum is over the set $\{(\nu,r) \in \{1,\dots,k\}\times\{0,1,\dots,k\}: \nu+r=k\}$. Using these last two facts, (4.21) and summing in a different order we obtain

$$(4.22)\qquad \Psi_p = I_p^\sigma(f_p) + \sum_{\nu=1}^{[\frac{p}{2}]} \sum_{r=0}^{\frac{[p-2\nu]}{2}} \frac{(-1)^{\nu+1}p!\sigma^{2\nu+2r}}{(p-2\nu-2r)!2^{\nu+r}\nu!r!} I^\sigma_{p-2\nu-2r}(Tr^{\nu+r} f_p)$$

$$= I_p^\sigma(f_p) + \sum_{k=1}^{[\frac{p}{2}]} \frac{p!\sigma^{2k}}{(p-2k)!2^k} \{\sum \frac{(-1)^{\nu+1}}{\nu!r!}\} I^\sigma_{p-2k}(Tr^k f_p)$$

$$= I_p^\sigma(f_p) + \sum_{k=1}^{[\frac{p}{2}]} \frac{p!\sigma^{2k}}{(p-2k)!2^k k!} I^\sigma_{p-2k}(Tr^k f_p)$$

$$= \sum_{k=0}^{[\frac{p}{2}]} \sigma^{2k} C_{p,k} I^\sigma_{p-2k}(Tr^k f_p),$$

and this proof is complete. □

5. LIFTINGS OF P-FORMS AND WIENER-ITO INTEGRALS. Let $f_p \in L_s^2(\mathbb{R}_+^p)$. We show in the theorem below that the p-form $\psi_p = \psi_p(f_p)$ on $L^2(\mathbb{R}_+)$ associated with f_p (see (4.2)) has a scaled $\mathcal{L}^2$-lifting $R[\psi_p]$ if and only if $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\ldots,[p/2]$. Further, in this case, $R[\psi_p]$ is given by a p-form on Wiener space or, alternately, by a finite sum of multiple Wiener-Itô integrals. This result is the key to Sections 5 through 8. Our proof that the trace conditions are necessary as well as sufficient for the existence of the lifting is tied in with the nature of the limiting k-trace.

Not only will the theorem below allow us to give in subsequent sections a solution to the problem which originally motivated our research, but it is potentially a useful result in connection with white noise calculus [13] where p-forms play a role analogous to that played by pth homogeneous chaos in Wiener calculus.

Theorem 5.1. Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H=L^2(\mathbb{R}_+)$.

$\psi_p(f_p)$ has a scaled $\mathcal{L}^2$-lifting $R[\psi_p]$ if and only if $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\ldots,[p/2]$. In this case, s-a.s. (that is, for every $\sigma > 0$, P_σ-a.s. on Ω_σ) we have

$$R\psi_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\xi_{i_1}\cdot\ldots\cdot\xi_{i_p} \tag{5.1}$$

$$= \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I_{p-2k}^{\sigma}(\overrightarrow{\mathrm{Tr}}^k f_p)$$

where the second expression in (5.1) is to be interpreted on each Ω_σ as the limit in the space $L^2(\Omega_\sigma, P_\sigma)$ of the sequence $\sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\xi_{i_1}\cdot\ldots\cdot\xi_{i_p}$.

Remark 5.1. The third expression in (5.1) has the advantage that it is coordinate free but the disadvantage that it must be changed with each change

in σ. The lifting Rf is, of course, scale-invariant measurable as well as s-a.s. defined.

Proof of Theorem 5.1. We begin with considerations which are relevant to both directions of the proof. Let $\{\pi_N\}$ be a sequence from $\mathcal{P}$, the class of orthogonal projections with finite-dimensional range, such that $\Pi_N \uparrow I$ strongly and $\dim(\Pi_N H)=d_N$. We can then obtain a CONS (ϕ_i) in H such that $\{\phi_i: i=1,\ldots,d_N\}$ is a CONS in $\Pi_N H$, N=1,2,... . It suffices to consider the case where $d_N = N$. Of course, f_p has an expansion with respect to the CONS $(\phi_{i_1}\otimes\ldots\otimes\phi_{i_p})$ for $L^2(\mathbb{R}_+)$:

$$f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}. \tag{5.2}$$

Now

$$\psi_p(h) = (f_p, h^{\otimes p}) = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\phi_{i_1},h)\cdot\ldots\cdot(\phi_{i_p},h), \tag{5.3}$$

and so

$$\begin{aligned}\psi_p\circ\Pi_N(h) = \psi_p(\Pi_N h) &= \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\phi_{i_1},\Pi_N h)\cdot\ldots\cdot(\phi_{i_p},\Pi_N h) \\ &= \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p}(\Pi_N\phi_{i_1},h)\cdot\ldots\cdot(\Pi_N\phi_{i_p},h) \\ &= \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}(\phi_{i_1},h)\cdot\ldots\cdot(\phi_{i_p},h) \\ &= (f_p^N, h^{\otimes p}),\end{aligned} \tag{5.4}$$

where f_p^N ($f^N_{p,(\phi_i)}$, more fully) satisfies

$$f_p^N = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_1}\otimes\ldots\otimes\phi_{i_p}. \tag{5.5}$$

It is easy to see from (5.4) how to lift the cylinder function $\psi_p \circ \Pi_N$:

$$R(\psi_p \circ \Pi_N) = \sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p} \xi_{i_1} \cdot \ldots \cdot \xi_{i_p} \tag{5.6}$$

where $\xi_i(x) = I_1(\phi_i)(x)$.

Applying Lemma 4.3 we see from (5.6) that for every $\sigma > 0$, P_σ-a.s. on Ω_σ,

$$R(\psi_p \circ \Pi_N) = \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^{\sigma}_{p-2k}(\mathrm{Tr}^k f_p^N). \tag{5.7}$$

Now suppose that $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\dots,[p/2]$. Let $\sigma > 0$ be given. We wish to show that

$$\| R(\psi_p \circ \Pi_N) - \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^{\sigma}_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p) \|^2_{L^2(\Omega_\sigma, P_\sigma)} \longrightarrow 0 \tag{5.8}$$

as $N \longrightarrow \infty$. However, using (5.7) and properties of multiple Wiener-Itô integrals (see (2.17)), we can write

$$\begin{aligned} &\| R(\psi_p \circ \Pi_N) - \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^{\sigma}_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p) \|^2_{L^2(\Omega_\sigma, P_\sigma)} \\ &= \| \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^{\sigma}_{p-2k}(\mathrm{Tr}^k f_p^N - \overrightarrow{\mathrm{Tr}}^k f_p) \|^2_{L^2(\Omega_\sigma, P_\sigma)} \qquad (5.9) \\ &= \sum_{k=0}^{[p/2]} (\sigma^{2k} C_{p,k})^2 (p-2k)! (\sigma^{p-2k})^2 \| \mathrm{Tr}^k f_p^N - \overrightarrow{\mathrm{Tr}}^k f_p \|^2_{L^2(\mathbb{R}_+^{p-2k})} \end{aligned}$$

but this last expression converges to 0 as $N \longrightarrow \infty$ by definition of $\overrightarrow{\mathrm{Tr}}^k f_p$, $k=0,1,\dots,[p/2]$.

We now know that ψ_p has a scaled $\mathcal{L}^2$-lifting $R\psi_p$ that is equal to the third expression in (5.1). Checking (5.7), we see that for every $\sigma > 0$ all three functions in (5.1) are equal P_σ-a.s. on Ω_σ.

It remains to show that if $\psi_p(f_p)$ has a scaled $\mathcal{L}^2$-lifting, then $\overrightarrow{\mathrm{Tr}}^k f_p$

exists for $k=0,1,\dots,[p/2]$. In fact, we will show that if there exists $\sigma>0$ such that $\psi_p(f_p)$ has a σ-$\mathcal{L}^2$-lifting, then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\dots,[p/2]$.

Let (α_i) be an arbitrary CONS for $H=L^2(\mathbb{R}_+)$ and let Π_N be the orthogonal projection onto $sp[\alpha_1,\dots,\alpha_N]$. By assumption, we have

$$E_{P_\sigma}[R(\psi_p\circ\Pi_N) - R\psi_p]^2 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty. \tag{5.10}$$

It follows that

$$E_{P_\sigma}[R(\psi_p\circ\Pi_N) - R(\psi_p\circ\Pi_{N'})]^2 \longrightarrow 0 \quad \text{as } N,N' \longrightarrow \infty. \tag{5.11}$$

But by Lemma 4.3 and (5.5) - (5.7),

$$R(\psi_p\circ\Pi_N) - R(\psi_p\circ\Pi_{N'}) = \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^\sigma_{p-2k}(\mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^{N'}_{p,(\alpha_i)}) \tag{5.12}$$

where

$$f^M_{p,(\alpha_i)} = \sum_{i_1,\dots,i_p=1}^{M} (f_p,\alpha_{i_1}\otimes\dots\otimes\alpha_{i_p})\alpha_{i_1}\otimes\dots\otimes\alpha_{i_p}. \tag{5.13}$$

Using properties of multiple Wiener-Itô integrals (see (2.17)), we have

$$\begin{aligned}(5.14)\ & E_{P_\sigma}[R(\psi_p\circ\Pi_N) - R(\psi_p\circ\Pi_{N'})]^2 \\ &= \sum_{k=0}^{[p/2]} (\sigma^{2k} C_{p,k})^2 (p-2k)!\, \| \mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^{N'}_{p,(\alpha_i)} \|_2^2 .\end{aligned}$$

From (5.14) and (5.11), we see that

$$\| \mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^{N'}_{p,(\alpha_i)} \|^2 \longrightarrow 0 \quad \text{as } N,N' \longrightarrow \infty \tag{5.15}$$

for $k=0,1,\dots,[p/2]$. Because of the completeness of $L^2(\mathbb{R}_+^{p-2k})$, (5.15) shows that, for every CONS (α_i), $\overrightarrow{\mathrm{Tr}}^k f^N_{p,(\alpha_i)}$ converges to an element of $L^2(\mathbb{R}_+^{p-2k})$. We still need to show that this limit is independent of the CONS (α_i).

Let (β_i) be any other CONS for H. It suffices to show that

(5.16) $$\|\mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^N_{p,(\beta_i)}\|_2 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

However,

(5.17) $$R(\psi_p \circ \Pi_{N,(\alpha_i)}) - R(\psi_p \circ \Pi_{N,(\beta_i)})$$
$$= \sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^\sigma_{p-2k}(\mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^N_{p,(\beta_i)}).$$

and so

(5.18) $$\sum_{k=0}^{[p/2]} \sigma^{2p}(C_{p,k})^2 (p-2k)!\,\|\mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^N_{p,(\beta_i)}\|_2^2$$
$$= E_{P_\sigma}\Big[\sum_{k=0}^{[p/2]} \sigma^{2k} C_{p,k} I^\sigma_{p-2k}(\mathrm{Tr}^k f^N_{p,(\alpha_i)} - \mathrm{Tr}^k f^N_{p,(\beta_i)})\Big]^2$$
$$= E_{P_\sigma}[R(\psi_p \circ \Pi_{N,(\alpha_i)}) - R(\psi_p \circ \Pi_{N,(\beta_i)})]^2.$$

But the last expression in (5.18) goes to 0 as $N \longrightarrow \infty$ since

$$E_{P_\sigma}[R(\psi_p \circ \Pi_{N,(\alpha_i)}) - R(\psi_p)]^2 \text{ and } E_{P_\sigma}[R(\psi_p \circ \Pi_{N,(\beta_i)}) - R(\psi_p)]^2$$

both go to 0 as $N \longrightarrow \infty$. It now follows from (5.18) that (5.16) holds for $k=0,1,\ldots,[p/2]$ as desired.

Hence, for every $k = 0,1,\ldots,[p/2]$, there exists g_{p-2k} in $L^2(\mathbb{R}_+^{p-2k})$ such that for every CONS (α_i) for H,

(5.19) $$\|\mathrm{Tr}^k f^N_{p,(\alpha_i)} - g_{p-2k}\|_2 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

Therefore $\overline{\mathrm{Tr}}^k f_p$ exists as we wished to show. □

Next we present some simple consequences of Theorem 5.1. (Much of the rest of this paper will give further consequences of this theorem.) The first corollary was already noted and established in the proof above but it seems worth stating formally.

Corollary 5.1. Let $f_p \in L^2_s(\mathbb{R}^p_+)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H=L^2(\mathbb{R}_+)$. If there exists $\sigma > 0$ such that ψ_p has a σ-$\mathcal{L}^2$-lifting, then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k = 0,1,\dots,[p/2]$. In particular, if ψ_p has an $\mathcal{L}^2$-lifting with respect to the standard Wiener measure $P=P_1$, then $\overrightarrow{\mathrm{Tr}}^k f_p$ exists for $k=0,1,\dots,[p/2]$.

Corollary 5.2. Let f_p and ψ_p be as in Corollary 5.1. Then ψ_p has a scaled $\mathcal{L}^2$-lifting if and only if there exists $\sigma > 0$ such that ψ_p has a σ-$\mathcal{L}^2$-lifting. In particular, if ψ_p has an $\mathcal{L}^2$-lifting with respect to the standard Wiener measure $P = P_1$, then ψ_p has a scaled L^2-lifting.

Remark 5.2. Comparing the first two expressions in (5.1) with (5.3), one sees that the series

$$(5.20)\qquad \sum_{i_1,\dots,i_p=1}^{\infty} a_{i_1,\dots,i_p}\,\xi_{i_1}\cdot\ldots\cdot\xi_{i_p} = \lim_{N\to\infty}\sum_{i_1,\dots,i_p=1}^{N} a_{i_1,\dots,i_p}\,\xi_{i_1}\cdot\ldots\cdot\xi_{i_p}$$

lifts ψ_p in a natural way.

Corollary 5.3. Let f_p and f'_p be in $L^2_s(\mathbb{R}^p_+)$ with associated p-forms ψ_p and ψ'_p, respectively. Suppose further that there exists $\sigma_o>0$ such that $R(\psi_p)$ and $R(\psi'_p)$ both exist in the σ_o-$\mathcal{L}^2$ sense (and hence in the scaled-$\mathcal{L}^2$ sense).

Then, for every $\sigma > 0$,

$$(5.21)\qquad \|R(\psi_p)-R(\psi'_p)\|^2_{L^2(\Omega_\sigma,P_\sigma)} = \sum_{k=0}^{[p/2]} (\sigma^{2p} C_{p,k})^2 (p-2k)!\,\|\overrightarrow{\mathrm{Tr}}^k f_p - \overrightarrow{\mathrm{Tr}}^k f'_p\|^2_{L^2(\mathbb{R}^{p-2k}_+)}.$$

In particular, if f_p and $f_p^{(n)}$, $n=1,2,\dots$, all belong to $L^2_s(\mathbb{R}^p_+)$ and if $\psi_p, \psi_p^{(n)}$, $n=1,2,\dots$ are the associated p-forms all of which possess scaled $\mathcal{L}^2$-liftings $R(\psi_p)$, $R(\psi_p^{(n)})$, $n=1,2,\dots$, then

$$(5.22)\qquad \|R(\psi_p)-R(\psi'_p)\|^2_{L^2(\Omega_\sigma,P_\sigma)} \longrightarrow 0 \quad \text{as } n\longrightarrow\infty$$

for every $\sigma > 0$ if and only if

$$\|\overrightarrow{\mathrm{Tr}}^{k} f_p^{(n)} - \overrightarrow{\mathrm{Tr}}^{k} f_p\|_{L_s^2(\mathbb{R}_+^{p-2k})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for each $k=0,1,\ldots,[p/2]$.

Proof. This follows easily from (5.1) and properties of multiple Wiener-Itô integrals. □

Remark 5.3. Often, in order to work with functions on white noise space, that is on the canonical Hilbert space, $(H,\mathscr{C},\mu)$, one must lift them to an associated countably additive representation space. It is sometimes possible to work directly in the Hilbert space. Corollary 5.3 allows us to do that for p-forms on $H=L^2(\mathbb{R}_+^p)$; the $L^2(C_0(\mathbb{R}_+),P_\sigma)$-distance from $R(\psi_p)$ to $R(\psi_p')$ can be calculated within the Hilbert space.

6. WIENER-ITO INTEGRALS IN TERMS OF LIFTED P-FORMS. The theorem of this section will show how to write the multiple Wiener-Itô integral $I_p^\sigma(f_p)$ as a finite sum of lifted (p-2k)-forms, k=0,1,...,[p/2]. The $\sigma=1$ case of this result will lead us in the next section to our definition of the natural extension $N[I_p(f_p)]$ of $I_p(f_p)$. At this point the reader may wish to review Definition 3.3 and Propositions 3.4 and 3.5.

Theorem 6.1. Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H=L^2(\mathbb{R}_+)$. Assume that f_p has all its first and second order traces and that they are consistent.

Then for every $\sigma > 0$, P_σ-a.s. on Ω_σ,

$$(6.1)\qquad I_p^\sigma(f_p) = \sum_{k=0}^{[p/2]} (-1)^k \sigma^{2k} C_{p,k} R[\psi_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p)]$$

$$= \sum_{k=0}^{[p/2]} (-1)^k \sigma^{2k} C_{p,k} \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \cdot \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}$$

where $R[\psi_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p)]$ is the lifting of the (p-2k)-form ψ_{p-2k} associated with the function $\overrightarrow{\mathrm{Tr}}^k f_p$ in $L_s^2(\mathbb{R}_+^{p-2k})$ and where (ϕ_i) is a CONS for $L^2(\mathbb{R}_+)$, $f_p = \sum_{i_1,\ldots,i_p=1}^{\infty} a_{i_1,\ldots,i_p} \phi_{i_1} \otimes \ldots \otimes \phi_{i_p}$, $\xi_i = I_1(\phi_i)$, and the sum over the i's in the third expression in (6.1) is interpreted on each Ω_σ as the limit in the space $L^2(\Omega_\sigma, P_\sigma)$ of the sequence

$$\sum_{i_{2k+1},\ldots,i_p=1}^{N} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p}.$$

Proof. We begin by establishing the first equality in (6.1). Our assumptions assure us, by Theorem 5.1, that the second expression in (6.1) makes sense and that formula (5.1) can be applied to each function $R[\psi_{p-2k}(\overrightarrow{\mathrm{Tr}}^k f_p)]$ in the

second expression in (6.1). Our first equality below is obtained by doing this and using the consistency of the first and second order traces. The second equality comes simply from splitting off the $k=0$, $\nu=0$ term and the third equality from doing the remaining sum in a different order. Finally, the fourth equality comes from the Binomial Theorem;

$$0 = [1+(-1)]^r = r! \sum_{k+\nu=r} \frac{(-1)^k}{k!\nu!} .$$

We now carry out the steps which were just commented on above:

$$(6.2) \qquad \sum_{k=0}^{[p/2]} (-1)^k \sigma^{2k} C_{p,k} R[\psi_{p-2k}(\overrightarrow{\mathrm{Tr}}^{k} f_p)]$$

$$= \sum_{k=0}^{[p/2]} \frac{(-1)^k \sigma^{2k} p!}{(p-2k)!2^k k!} \sum_{\nu=0}^{[\frac{p-2k}{2}]} \frac{(p-2k)!\sigma^{2\nu}}{(p-2k-2\nu)!2^\nu v!} I^\sigma_{p-2k-2\nu}(\overrightarrow{\mathrm{Tr}}^{k+\nu} f_p)$$

$$= I^\sigma_p(f_p) + \sum_{\substack{k=0 \\ (k,\nu)\neq(0,0)}}^{[p/2]} \sum_{\nu=0}^{[\frac{p-2k}{2}]} \frac{(-1)^k p! \sigma^{2(k+\nu)}}{(p-2k-2\nu)!2^{k+\nu} k!\nu!} I^\sigma_{p-2k-2\nu} (\overrightarrow{\mathrm{Tr}}^{k+\nu} f_p)$$

$$= I^\sigma_p(f_p) + \sum_{r=1}^{[p/2]} \frac{p!\sigma^{2r}}{(p-2r)!2^r} \Big\{ \sum_{k+\nu=r} \frac{(-1)^k}{k!\nu!} \Big\} I^\sigma_{p-2r}(\overrightarrow{\mathrm{Tr}}^{r} f_p)$$

$$= I^\sigma_p(f_p)$$

as claimed.

It remains to establish the second equality in (6.1). By Proposition 3.3,

$$(6.3) \quad \overrightarrow{\mathrm{Tr}}^{k} f_p = \sum_{i_{2k+1},\dots,i_p=1}^{\infty} \Big(\sum_{j_1,\dots,j_k=1}^{\infty} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big) \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}$$

so that

$$(6.4) \quad \psi_{p-2k}(\overrightarrow{\mathrm{Tr}}^{k} f_p)(h) = \sum_{i_{2k+1},\dots,i_p=1}^{\infty} \Big(\sum_{j_1,\dots,j_k=1}^{\infty} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p} \Big)$$

$$\cdot\ (\phi_{i_{2k+1}},h)\cdot\ldots\cdot(\phi_{i_p},h)$$

and so by Theorem 5.1,

(6.5) $$R[\psi_{p-2k}(\overline{\mathrm{Tr}}^{k} f_p)]$$

$$= \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \xi_{i_{2k+1}} \cdot\ldots\cdot \xi_{i_p} . \qquad \square$$

7. <u>THE NATURAL EXTENSION OF A WIENER-ITO INTEGRAL.</u> The case $\sigma=1$ of Theorem 6.1 and the idea that the natural extension should preserve polynomials in the σ_i's immediately suggests how to define $N[I_p(f_p)]$.

<u>Definition 7.1.</u> The <u>natural extension</u> of $I_p(f_p)$ is defined by

$$(7.1)\qquad N[I_p(f_p)] = \sum_{k=0}^{[p/2]} (-1)^k C_{p,k} R[\psi_{p-2k}(\overrightarrow{Tr}^{k} f_p)] = \sum_{k=0}^{[p/2]} (-1)^k C_{p,k} \sum_{i_{2k+1},\ldots,i_p=1}^{\infty} \Big(\sum_{j_1,\ldots,j_k=1}^{\infty} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big) \xi_{i_{2k+1}} \cdot \ldots \cdot \xi_{i_p} .$$

The next theorem gives us conditions under which Definition 7.1 makes sense.

<u>Theorem 7.1.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H=L^2(\mathbb{R}_+)$. Assume that f_p has all its first and second order traces and that they are consistent.

Then the last two expressions in formula (7.1) are defined and agree s-a.s. and are scale-invariant measurable.

<u>Proof.</u> Apply Theorem 5.1 and Proposition 3.3 to each of the (p-2k)-forms $\psi_{p-2k}(\overrightarrow{Tr}^{k} f_p)$, $k=0,1,\ldots,[p/2]$. □

Next we give a theorem which will express the restriction of $N[I_p(f_p)]$ to Ω_σ as a sum of multiple Wiener-Itô integrals $I^{\sigma}_{p-2k}(\overrightarrow{Tr}^{k} f_p)$, $k=0,1,\ldots,[p/2]$. This theorem will be the basis of our calculation in the next section of the analytic Feynman integral of $N[I_p(f_p)]$.

<u>Theorem 7.2.</u> Let $f_p \in L_s^2(\mathbb{R}_+^p)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H = L^2(\mathbb{R}_+)$. Assume that f_p has all its first and second order traces and that they are consistent.

Then, for any $\sigma > 0$, we have P_σ-a.s. on Ω_σ,

$$(7.2) \qquad N[I_p(f_p)] = \sum_{k=0}^{[p/2]} (\sigma^2-1)^k C_{p,k} I^{\sigma}_{p-2k}(\overrightarrow{\mathrm{Tr}}^{k} f_p).$$

Proof. Proposition 3.4 allows us to apply Theorem 6.1 to the right-hand side of (7.2); this yields the first equality below. The third equality comes from reordering the sum and the fourth from applying the Binomial Theorem to $(-1)^r=[(\sigma^2-1)+(-\sigma^2)]^r$. Finally the last equality comes from Definition 7.1 and Theorem 7.1.

$$(7.3) \quad \sum_{k=0}^{[p/2]} (\sigma^2-1)^k C_{p,k} I^{\sigma}_{p-2k}(\overrightarrow{\mathrm{Tr}}^{k} f_p)$$

$$= \sum_{k=0}^{[p/2]} \frac{p!(\sigma^2-1)^k}{(p-2k)!2^k k!} \left\{ \sum_{\nu=0}^{[\frac{p-2k}{2}]} \frac{(p-2k)!(-1)^{\nu}\sigma^{2\nu}}{(p-2k-2\nu)!2^{\nu}\nu!} R[\psi_{p-2k-2\nu}(\overrightarrow{\mathrm{Tr}}^{k+\nu} f_p)] \right\}$$

$$= \sum_{k=0}^{[p/2]} \sum_{\nu=0}^{[\frac{(p-2k)}{2}]} \frac{p!(\sigma^2-1)^k(-1)^{\nu}\sigma^{2\nu}}{(p-2k-2\nu)!2^{k+\nu}k!\nu!} R[\psi_{p-2k-2\nu}(\overrightarrow{\mathrm{Tr}}^{k+\nu} f_p)]$$

$$= \sum_{r=0}^{[p/2]} \frac{p!}{(p-2r)!2^r} \left\{ \sum_{k+\nu=r} \frac{(\sigma^2-1)^k(-\sigma^2)^{\nu}}{k!\nu!} \right\} R[\psi_{p-2r}(\overrightarrow{\mathrm{Tr}}^{r} f_p)]$$

$$= \sum_{r=0}^{[p/2]} (-1)^r C_{p,r} R[\psi_{p-2r}(\overrightarrow{\mathrm{Tr}}^{r} f_p)]$$

$$= N[I_p(f_p)]$$

as desired. □

Remark 7.1. Formula (7.2) is closely related to (8) in [5], a key formula in the paper of Hu and Meyer. However, in [5], the right-hand side of (7.2) is their "suggested definition" of the natural extension of $I_p(f_p)$ to Ω_{σ}. Hu and Meyer do not have one formula (like our (7.1)) which extends $I_p(f_p)$ to Ω_{σ} for all $\sigma > 0$. Their "suggested definition" has become a theorem in our

development. Further, our Definition 7.1 reflects in a rather transparent way the desire to extend $I_p(f_p)$ so as to preserve polynomials in the ξ_i's.

Next we give a simple example which will illustrate our formulas and help to clairfy the idea of the natural extension and the concept of functions equal s-a.s. Let $\phi := \chi_{[0,1]}$ so that $f_1 := \phi \otimes \phi$ belongs to $L_s^2(\mathbb{R}_+^2)$. We introduce a function G: $\mathscr{C}_0(\mathbb{R}_+) \longrightarrow \mathbb{R}$ which is everywhere defined and continuous and with which we will compare two different extensions of $I_2(f_2)$: $\Omega_1 \longrightarrow \mathbb{R}$. Let $G(x) := x^2(1)-1$.

Using the third expression in (6.1) (or (4.9), the corresponding formula for finite expansions) with $\sigma = 1$, $p = 2$, $k = 0,1$, we see that P_1-a.s. on Ω_1,

$$I_2(f_2)(x) = \xi^2(x)-1 = [\int_{\mathbb{R}_+} \phi(s)dx(s)]^2 - 1$$

$$= [\int_0^1 dx(s)]^2 - 1 = x^2(1) - 1$$

where $\xi = I_1(\phi)$. Hence $I_2(f_2) = G$ P_1-a.s. on Ω_1.

By (7.1), $N[I_2(f_2)] = \xi^2-1$. In particular, for every $\sigma > 0$, P_1-a.s. on Ω_1,

$$N[I_2(f_2)](\sigma x) = \xi^2(\sigma x)-1 = \sigma^2\xi^2(x)-1 = \sigma^2x^2(1)-1.$$

We can also obtain the formula just above by using (7.2) of Theorem 7.2: For every $\sigma > 0$, P_1-a.s. on Ω_1,

$$N[I_2(f_2)](\sigma x) = C_{2,0}I_2^\sigma(\overrightarrow{\mathrm{Tr}}^0 f_2)(\sigma x) + (\sigma^2-1)C_{2,1}I_0^\sigma(\overrightarrow{\mathrm{Tr}}^1 f_2)(x)$$

$$= I_2^\sigma(f_2)(\sigma x) + (\sigma^2-1)\overrightarrow{\mathrm{Tr}}^1 f_2 = \sigma^2 I_2(f_2)(x) + (\sigma^2-1)$$

$$= \sigma^2[\xi^2(x)-1] + \sigma^2-1 = \sigma^2\xi^2(x)-1 = \sigma^2x^2(1)-1.$$

Note that $N[I_2(f_2)] = G$ s-a.s.

In contrast, we define the extension F of $I_2(f_2)$ to $\bigcup_{\sigma>0} \Omega_\sigma$ by $F(\sigma x) = I_2^\sigma(f_2)(\sigma x)$. Now P_1-a.s. on Ω_1, $F(\sigma x) = \sigma^2 I_2(f_2)(x) = \sigma^2[\xi^2(x)-1] = \sigma^2 x^2(1)-\sigma^2$.

Summarizing the above: $N[I_2(f_2)] = G$ s-a.s. and $N[I_2(f_2)] = G = F$ P_1-a.s. However, for every $\sigma \neq 1$, it is not true that $F = G$ P_σ-a.s., and, in fact,

$$P_\sigma\{y\in\Omega_\sigma: F(y) \neq G(y)\} = P_1\{x\in\Omega_1: F(\sigma x) \neq G(\sigma x)\} = 1$$

since $G(\sigma x) = \sigma^2 x^2(1)-1$ for every $x \in \Omega_1$ whereas $F(\sigma x) = \sigma^2 x^2(1)-\sigma^2$ P_1-a.s. on Ω_1.

We finish this section by discussing briefly a generalization of the results above. This generalization is related to a further formula of Hu and Meyer [5]. Let $f_p \in L_s^2(\mathbb{R}_+^p)$. We will define the natural extension $N[I_p^\tau(f_p)]$ for any $\tau > 0$. Again, Theorem 6.1 suggests how this should be done.

Definition 7.2. The natural extension of $I_p^\tau(f_p)$ is defined by

$$(7.4)\qquad N[I_p^\tau(f_p)] = \sum_{k=0}^{[p/2]} (-1)^k \tau^{2k} C_{p,k} R[\psi_{p-2k}(\overrightarrow{Tr}^k f_p)]$$

$$= \sum_{k=0}^{[p/2]} (-1)^k \tau^{2k} C_{p,k} \sum_{i_{2k+1},\dots,i_p=1}^{\infty}$$

$$\Big(\sum_{j_1,\dots,j_k=1}^{\infty} a_{j_1,j_1,\dots,j_k,j_k,i_{2k+1},\dots,i_p}\Big)\xi_{i_{2k+1}}\cdot\ldots\cdot\xi_{i_p}.$$

Under the assumption that f_p has all its first and second order traces and they are consistent, we can show just as in Theorem 7.1 that Definition 7.2 makes sense. Next we ask for a formula extending (7.2).

Theorem 7.3. Under the assumption of Theorem 7.2, for any $\sigma > 0$, we have P_σ-a.s. on Ω_σ,

$$N[I_p^\tau(f_p)] = \sum_{k=0}^{[p/2]} (\sigma^2-\tau^2)^k C_{p,k} I_{p-2k}^\sigma(\overrightarrow{Tr}^k f_p). \tag{7.5}$$

<u>Proof.</u> The argument proceeds just as in the proof of Theorem 7.2 except that here one applies the Binomial Theorem to

$$[-\tau^2]^r = [(\sigma^2-\tau^2) + (-\sigma^2)]^r. \qquad \square$$

8. THE FEYNMAN INTEGRAL OF THE NATURAL EXTENSION OF A WIENER-ITO INTEGRAL.

Every square integrable function on Wiener space, that is, every f in $L^2(C_0(\mathbb{R}_+),P_1) = L^2(\Omega_1,P_1)$ has an expansion in Wiener chaos,

$$f = \sum_{p\geq 0} \frac{1}{p!} I_p(f_p), \tag{8.1}$$

where $f_p \in L^2_s(\mathbb{R}^p_+)$. Hu and Meyer have the following formula [5, (7)] in terms of the expansion (8.1):

$$E_\sigma(f) = \sum_k \frac{(\sigma^2-1)^k}{2^k k!} \mathrm{Tr}^k(f_{2k}). \tag{8.2}$$

The formula (8.2) is to give the "Feynman integral" of f when σ^2 is purely imaginary and the right-hand side of the formula makes sense. It is natural to ask, as Hu and Meyer did, what, if anything, this has to do with the idea of obtaining the Feynman integral by analytically continuing the Wiener integral $\int_{\mathscr{C}_0} f(\sigma x)dP_1(x)$. It would seem naively that for the two approaches to be consistent, one should have for any $\sigma > 0$

$$\int_{\mathscr{C}_0(\mathbb{R}_+)} \frac{1}{p!} I_p(f_p)(\sigma x)dP_1(x) = \begin{cases} \dfrac{(\sigma^2-1)^k}{2^k k!} \mathrm{Tr}^k(f_{2k}) & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases} \tag{8.3}$$

<u>However</u>, (8.3) <u>is too naive</u> since, P_1-a.s., σx is in Ω_σ and it is not clear a priori how to extend $I_p(f_p)$ from Ω_1 to Ω_σ. The theorem below will show that the natural extension $N[I_p(f_p)]$ described in the previous section produces the desired formula.

<u>Theorem 8.1.</u> Let $f_p \in L^2_s(\mathbb{R}^p_+)$ and let $\psi_p = \psi_p(f_p)$ be the associated p-form on $H = L^2(\mathbb{R}_+)$. Assume that f_p has all its first and second order traces and that they are consistent.

Then, for every $\sigma > 0$, we have the form

$$(8.4)\qquad \int_{\mathscr{C}_0(\mathbb{R}_+)} \frac{1}{p!} N[I_p(f_p)](\sigma x)dP_1(x) = \begin{cases} \dfrac{(\sigma^2-1)^k}{2^k k!} \overrightarrow{\mathrm{Tr}}^k(f_{2k}) & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Further, the analytic Feynman integral of the function $\frac{1}{p!} N[I_p(f_p)]$ is given by the right-hand side of (8.4) with $\sigma^2 = -i$.

Proof. Formula (7.2) of Theorem 7.2 is the key to the proof. Fix $\sigma > 0$. The steps below follow from (2.1), (7.2) and the fact that, for $p \geq 1$, $\int_{\mathscr{C}_0(\mathbb{R}_+)} I_p^\sigma(f_p)(x)dP_\sigma(x) = 0$:

$$\int_{\mathscr{C}_0(\mathbb{R}_+)} \frac{1}{p!} N[I_p(f_p)](\sigma x)dP_1(x) = \int_{\mathscr{C}_0(\mathbb{R}_+)} \frac{1}{p!} N[I_p(f_p)](x)dP_\sigma(x)$$

$$= \frac{1}{p!} \sum_{\nu=0}^{[p/2]} \frac{(\sigma^2-1)^\nu p!}{(p-2\nu)!2^\nu \nu!} \int_{\mathscr{C}_0(\mathbb{R}_+)} I_{p-2\nu}^\sigma(\overrightarrow{\mathrm{Tr}}^\nu f_p)(x)dP_\sigma(x)$$

$$= \begin{cases} \dfrac{(\sigma^2-1)^k}{2^k k!} \overrightarrow{\mathrm{Tr}}^k f_{2k} & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

which is the desired formula. □

Remark 8.1. It is natural to investigate the class of functions f which possess a Feynman integral in the sense of [5] and, in particular, to ask if various functions of interest in quantum mechanics fall in this class. Such matters are discussed to some extent in [5] and are of interest to the authors but they have not been our concern in this paper.

We finish this section by stating a result like Theorem 8.1 except that it involves the natural extension of $I_p^\tau(f_p)$ rather than $N[I_p(f_p)]$.

Theorem 8.2. Let the assumptions of Theorem 8.1 be satisfied and let

$\tau > 0$ be given. Then, for every $\sigma > 0$, we have the formula

$$(8.5)\qquad \int_{\mathscr{C}_0(\mathbb{R}_+)} \frac{1}{p!} N[I_p^{\tau}(f_p)](\sigma x)dP_1(x) = \begin{cases} \dfrac{(\sigma^2-\tau^2)^k}{2^k k!}\,\overline{\mathrm{Tr}}^k(f_{2k}) & \text{if } p=2k \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Proof. The proof is the same as the proof of Theorem 8.1 except that formula (7.5) is used instead of (7.2). □

Remark 8.2. If we analytically continue the right-hand side of (8.5) to $\sigma^2=-i$, we obtain the analytic Feynman integral of $\frac{1}{p!} N[I_p^{\tau}(f_p)]$.

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